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A DETERMINATION OF THE TYPES OF PLANAR CREMONA TRANSFORMATIONS WITH NOT MORE THAN 9 *F*-POINTS.

By Mildred E. Taylor.

1. *Introduction.* The types of Cremona transformations can be classified according to the order n or according to the number ρ of the F-points. Coble, [3, p. 83], remarks "that perhaps the latter classification is the more fundamental." Roughly speaking, the range of usefulness of a transformation T is inversely proportional to the number ρ of the F-points. For, unless the F-points of T can be placed at significant points of a given figure, the image is more complicated rather than more simple.

Cremona[6,7] gave a list of transformations for $n = 2$ to $n = 10$. Guccia[8] gave algebraic expressions for the types and their inverses as listed by Cremona. Mlodziejowski [10] listed types for $n = 2$ to $n = 21$. Bianchi [1] by multiplying two De Jonquières transformations obtained the expression for the order n involving 3 parameters. Palatini[12] by multiplying three De Jonquières transformations obtained a new transformation of order n whose expression involves 7 parameters.* He also multiplied four De Jonquières transformations to obtain a new one whose order n involves 15 parameters. In general, if k De Jonquières transformations be compounded the expression for the order n of the new transformation involves $(2^k - 1)$ parameters. Miss Hudson[9] gave types for $n = \alpha\mu - \beta$, where $\beta = 2\gamma + \epsilon$, $\epsilon = 0, 1$. If the number of F-points is 9, there exists an infinite number of transformations. However, in any of the above types when the number of F-points is limited to 9, only a finite number of transformations is obtained. Montesano[11] derived the semi-symmetric types for $\rho = 9$ and obtained independent expressions for each type involving 1 parameter. This paper shows that these types are all related.

In this paper I have obtained explicit algebraic expressions for the integers n, r_i, s_j, α_{ji} attached to every planar Cremona transformation with not more than 9-points, say at $P_9{}^2$, i. e. $p_1, \cdots, p_9$. The method is that used throughout the literature—the composition of known types. These algebraic expressions are classified into *seven* distinct types.

* Miss Hudson [9] states that the expression which Palatini obtained for n by compounding three De Jonquières transformations involves 6 parameters and by compounding four De Jonquières transformations involves 16 parameters.

2. *The Types in the Invariant Subgroup $i_{9,2}$ of the Linear Group $g_{9,2}$.* The product of two Bertini involutions E_2 and E_1 with simple points at p_2, p_1 respectively is called $C_{2,1}$. The semi-symmetric transformation of order 37 with eight 13-fold points and one 4-fold point is called D_9 when the 4-fold point is at p_9. The general element [5, p. 375] of the abelian subgroup of the invariant subgroup $i_{9,2}$ of the linear group $g_{9,2}$ is given by

$$(1) \qquad D_2^{\nu_2} D_3^{\nu_3} \cdots D_9^{\nu_9} C_{2,1}^{\rho_2} C_{3,1}^{\rho_3} \cdots C_{9,1}^{\rho_9},$$

where ν_i is any integer, positive or negative, and $\rho_i = 0, 1, 2$ with $\Sigma \rho_i \equiv 0 \bmod 3$ $(i = 2, 3, \cdots, 9)$.

By direct multiplication of the factors of (1), explicit algebraic expressions are obtained for n, r_i, s_j, α_{ji} of Type I.

The following notation is used throughout for the elementary symmetric functions of the ν's and the ρ's:

$$(2) \qquad \nu = \Sigma_2^9 \nu_i, \quad \nu' = \Sigma_2^9 \nu_i \nu_j; \quad \rho = \Sigma_2^9 \rho_i, \quad \rho' = \Sigma_2^9 \rho_i \rho_j; \quad \sigma = \Sigma_2^9 \nu_i \rho_i.$$

Also let

$$(3) \qquad \gamma = 4\nu^2 + 4\rho^2 - 9\nu' - 4\rho' - 6\sigma.$$

Type I.

$$D_2^{\nu_2} D_3^{\nu_3} \cdots D_9^{\nu_9} C_{2,1}^{\rho_2} C_{3,1}^{\rho_3} \cdots C_{9,1}^{\rho_9}.$$

With $i, j = 1, 2, \cdots, 9$ $(i \neq j)$ and $k = 2, 3, \cdots, 9$, let

$$\delta_1 = 2\rho, \quad \delta_k = 3\nu_k - 2\rho_k.$$

Then

$$(4) \qquad \begin{aligned} &n = 9\gamma + 1, \quad r_i = 3\gamma + \nu - 3\delta_i, \quad s_j = 3\gamma - \nu + 3\delta_j, \\ &\alpha_{ii} = \gamma - 1, \quad \alpha_{ji} = \gamma + \delta_j - \delta_i. \end{aligned}$$

The remaining elements of $i_{9,2}$ [5, p. 375] are obtained by taking the product of E_1 and (1).

Type II.

$$E_1 D_2^{\nu_2} D_3^{\nu_3} \cdots D_9^{\nu_9} C_{2,1}^{\rho_2} C_{3,1}^{\rho_3} \cdots C_{9,1}^{\rho_9}.$$

With i, j, k and δ_1, δ_k as in Type I let

$$\gamma' = \gamma - \nu + 4\rho + 2, \quad \delta'_1 = \delta_1 + 2, \quad \delta'_k = \delta_k.$$

Then

$$(5)\quad \begin{aligned} &n = 9\gamma' + 3\nu - 1, \quad r_i = 3\gamma' + 2\nu - 3\delta_i, \quad s_j = 3\gamma' + 2\nu - 3\delta'_j \\ &\alpha_{ii} = \gamma' + \nu - 2\delta'_i + 1, \quad \alpha_{ji} = \gamma' + \nu - \delta'_i - \delta'_j. \end{aligned}$$

3. *The Remaining Types in the Linear Group $g_{9,2}$.* The factor group $f_{9,2}$ of the group $g_{9,2}$ [5, p. 373] with respect to $i_{9,2}$ is simply isomorphic with the Cremona group $G_{8,2}$ [4, p. 15; 5, p. 349-350]. A study of the transformations in $G_{8,2}$, which is a finite group, shows that each one can be obtained from an element of $i_{9,2}$ by not more than two quadratic transformations. Each type of $i_{9,2}$ is multiplied by a single quadratic, two unrelated quadratic (no F-points in common), or two related quadratic transformations (one F-point in common). There are exactly three different elements of the abelian subgroup of $i_{9,2}$ that will give the same Cremona transformation when multiplied by two unrelated quadratic transformations. Exactly three elements of the non-abelian part of $i_{9,2}$ when multiplied by two unrelated quadratic transformations give the same Cremona transformation. A Cremona transformation which is the product of an element of the abelian subgroup of $i_{9,2}$ and two related quadratic transformations is also the product of some element of the non-abelian part of $i_{9,2}$ and two related quadratic transformations. Hence, there are only *seven* distinct types of planar Cremona transformations.

Since the F-points can be permuted without altering the transformation, the quadratic transformations A_{123}, A_{456}, A_{145} with F-points at p_1, p_2, p_3; p_4, p_5, p_6; p_1, p_4, p_5, respectively are used to simplify the notation.

Type III.

$$D_2^{\nu_2} D_3^{\nu_3} \cdots D_9^{\nu_9} C_{2,1}^{\rho_2} C_{3,1}^{\rho_3} \cdots C_{9,1}^{\rho_9} A_{123}.$$

With $i, j = 1, 2, 3$ $(i \neq j)$ and $k, l = 4, 5, \cdots, 9$ $(k \neq l)$, let

$$\epsilon_{123} = \nu - (\delta_1 + \delta_2 + \delta_3).$$

Then

$$(6)\quad \begin{aligned} &n = 9\gamma - 3\epsilon_{123} + 2, \\ &r_i = 3\gamma + \nu - 3\epsilon_{123} - 3\delta_i + 1, \quad r_k = 3\gamma + \nu - 3\delta_k, \\ &s_j = 3\gamma - \nu - \epsilon_{123} + 3\delta_j + 1, \quad s_l = 3\gamma - \nu - \epsilon_{123} + 3\delta_l, \\ &\alpha_{ii} = \gamma - \epsilon_{123}, \quad \alpha_{kk} = \gamma - 1, \quad \alpha_{ji} = \gamma - \epsilon_{123} + \delta_j - \delta_i + 1, \\ &\alpha_{jk} = \gamma - \epsilon_{123} + \delta_j - \delta_k, \quad \alpha_{li} = \gamma + \delta_l - \delta_i, \quad \alpha_{lk} = \gamma + \delta_l - \delta_k. \end{aligned}$$

Type IV.

$$E_1 D_2^{\nu_2} D_3^{\nu_3} \cdots D_9^{\nu_9} C_{2,1}^{\rho_2} C_{3,1}^{\rho_3} \cdots C_{9,1}^{\rho_9} A_{123}.$$

With γ' and δ''s as in Type II, and i, j, k, l as in Type III, let

$$\epsilon'_{123} = \nu - (\delta'_1 + \delta'_2 + \delta'_3).$$

Then

$$(7)\quad \begin{aligned}
&n = 9\gamma' + 3\nu - 3\epsilon'_{123} - 2,\\
&r_i = 3\gamma' + 2\nu - 3\epsilon'_{123} - 3\delta'_i - 1, \quad r_k = 3\gamma' + 2\nu - 3\delta'_k,\\
&s_j = 3\gamma' + 2\nu - \epsilon'_{123} - 3\delta'_j - 1, \quad s_l = 3\gamma' + 2\nu - \epsilon'_{123} - 3\delta'_l,\\
&\alpha_{ii} = \gamma' + \nu - \epsilon'_{123} - 2\delta'_i, \quad \alpha_{kk} = \gamma' + \nu - 2\delta'_k + 1,\\
&\alpha_{ji} = \gamma' + \nu - \epsilon'_{123} - \delta'_i - \delta'_j - 1, \quad \alpha_{li} = \gamma' + \nu - \epsilon'_{123} - \delta'_l - \delta'_i,\\
&\alpha_{jk} = \gamma' + \nu - \delta'_j - \delta'_k, \quad \alpha_{lk} = \gamma' + \nu - \delta'_l - \delta'_k.
\end{aligned}$$

Type V.

$$D_2{}^{\nu_2}D_3{}^{\nu_3}\cdots D_9{}^{\nu_9}C_{2,1}{}^{\rho_2}C_{3,1}{}^{\rho_3}\cdots C_{9,1}{}^{\rho_9}A_{123}A_{456}.$$

With $i, j = 1, 2, 3$ $(i \neq j)$, $k, l = 4, 5, 6$ $(k \neq l)$, and $m, t = 7, 8, 9$ $(m \neq t)$, let $\epsilon_{456} = \nu - (\delta_4 + \delta_5 + \delta_6)$. Then

$$(8)\quad \begin{aligned}
&n = 9\gamma - 6\epsilon_{123} - 3\epsilon_{456} + 4,\\
&r_i = 3\gamma + \nu - 3\epsilon_{123} - 3\delta_i + 1, \quad r_k = 3\gamma + \nu - 3\epsilon_{123} - 3\epsilon_{456} - 3\delta_k + 2,\\
&r_m = 3\gamma + \nu - 3\delta_m, \quad s_j = 3\gamma - \nu - 2\epsilon_{123} - \epsilon_{456} + 3\delta_j + 2,\\
&s_l = 3\gamma - \nu - 2\epsilon_{123} - \epsilon_{456} + 3\delta_l + 1, \quad s_t = 3\gamma - \nu - 2\epsilon_{123} - \epsilon_{456} + 3\delta_t,\\
&\alpha_{ii} = \gamma - \epsilon_{123}, \quad \alpha_{kk} = \gamma - \epsilon_{123} - \epsilon_{456}, \quad \alpha_{mm} = \gamma - 1,\\
&\alpha_{ji} = \gamma - \epsilon_{123} + \delta_j - \delta_i + 1, \quad \alpha_{li} = \gamma - \epsilon_{123} + \delta_k - \delta_i,\\
&\alpha_{ti} = \gamma - \epsilon_{123} + \delta_t - \delta_i, \quad \alpha_{jk} = \gamma - \epsilon_{123} - \epsilon_{456} + \delta_j - \delta_k + 1,\\
&\alpha_{lk} = \gamma - \epsilon_{123} - \epsilon_{456} + \delta_l - \delta_k + 1, \quad \alpha_{tk} = \gamma - \epsilon_{123} - \epsilon_{456} + \delta_t - \delta_k,\\
&\alpha_{jm} = \gamma + \delta_j - \delta_m, \quad \alpha_{lm} = \gamma + \delta_l - \delta_m, \quad \alpha_{tm} = \gamma + \delta_t - \delta_m.
\end{aligned}$$

Type VI.

$$E_1D_2{}^{\nu_2}D_3{}^{\nu_3}\cdots D_9{}^{\nu_9}C_{2,1}{}^{\rho_2}C_{3,1}{}^{\rho_3}\cdots C_{9,1}{}^{\rho_9}A_{123}A_{456}.$$

With i, j, k, l, m, t as in Type V and ϵ'_{123} as in Type IV, let

$$\epsilon'_{456} = \nu - (\delta'_4 + \delta'_5 + \delta'_6).$$

Then

$$(9)\quad \begin{aligned}
&n = 9\gamma' + 3\nu - 6\epsilon'_{123} - 3\epsilon'_{456} - 4,\\
&r_i = 3\gamma' + 2\nu - 3\epsilon'_{123} - 3\delta'_i - 1, \quad r_k = 3\gamma' + 2\nu - 3\epsilon'_{123} - 3\epsilon'_{456} - 3\delta'_k - 2,\\
&r_m = 3\gamma' + 2\nu - 3\delta'_m, \quad s_j = 3\gamma' + 2\nu - 2\epsilon'_{123} - \epsilon'_{456} - 3\delta'_j - 2,\\
&s_l = 3\gamma' + 2\nu - 2\epsilon'_{123} - \epsilon'_{456} - 3\delta'_l - 1, \quad s_t = 3\gamma' + 2\nu - 2\epsilon'_{123} - \epsilon'_{456} - 3\delta'_t,\\
&\alpha_{ii} = \gamma' + \nu - \epsilon'_{123} - 2\delta'_i, \quad \alpha_{kk} = \gamma' + \nu - \epsilon'_{123} - \epsilon'_{456} - 2\delta'_k,\\
&\alpha_{mm} = \gamma' + \nu - 2\delta'_m + 1, \quad \alpha_{ji} = \gamma' + \nu - \epsilon'_{123} - \delta'_i - \delta'_j - 1,\\
&\alpha_{li} = \gamma' + \nu - \epsilon'_{123} - \delta'_l - \delta'_i - 1, \quad \alpha_{ti} = \gamma' + \nu - \epsilon'_{123} - \delta'_t - \delta'_i - 1,
\end{aligned}$$

$$\alpha_{jk} = \gamma' + \nu - \epsilon'_{123} - \epsilon'_{456} - \delta'_j - \delta'_k - 1,$$
$$\alpha_{lk} = \gamma' + \nu - \epsilon'_{123} - \epsilon'_{456} - \delta'_l - \delta'_k - 1,$$
$$\alpha_{tk} = \gamma' + \nu - \epsilon'_{123} - \epsilon'_{456} - \delta'_t - \delta'_k, \quad \alpha_{jm} = \gamma' + \nu - \delta'_j - \delta'_m,$$
$$\alpha_{lm} = \gamma' + \nu - \delta'_l - \delta'_m, \quad \alpha_{tm} = \gamma' + \nu - \delta'_t - \delta'_m.$$

Type VII.

$$D_2^{\nu_2} D_3^{\nu_3} \cdots D_9^{\nu_9} C_{2,1}^{\rho_2} C_{3,1}^{\rho_3} \cdots C_{9,1}^{\rho_9} A_{123} A_{145}.$$

With $i, j = 2, 3$ $(i \neq j)$, $k, l = 4, 5$ $(k \neq l)$ and $m, t = 6, 7, 8, 9$ $(m \neq t)$ and ϵ_{123} as in Type III, let $\epsilon_{145} = \nu - (\delta_1 + \delta_4 + \delta_5)$. Then

$$\begin{aligned}
&n = 9\gamma - 3\epsilon_{123} - 3\epsilon_{145} + 3,\\
&r_1 = 3\gamma + \nu - 3\epsilon_{123} - 3\epsilon_{145} - 3\delta_1 + 2, \quad r_i = 3\gamma + \nu - 3\epsilon_{123} - 3\delta_i + 1,\\
&r_k = 3\gamma + \nu - 3\epsilon_{145} - 3\delta_k + 1, \quad r_m = 3\gamma + \nu - 3\delta_m,\\
&s_1 = 3\gamma - \nu - \epsilon_{123} - \epsilon_{145} + 3\delta_1 + 2, \quad s_j = 3\gamma - \nu - \epsilon_{123} - \epsilon_{145} + 3\delta_j + 1,\\
&s_l = 3\gamma - \nu - \epsilon_{123} - \epsilon_{145} + 3\delta_l + 1, \quad s_t = 3\gamma - \nu - \epsilon_{123} - \epsilon_{145} + 3\delta_t,\\
&\alpha_{11} = \gamma - \epsilon_{123} - \epsilon_{145} + 1, \quad \alpha_{ii} = \gamma - \epsilon_{123}, \quad \alpha_{kk} = \gamma - \epsilon_{145}, \quad \alpha_{mm} = \gamma - 1,\\
(10)\quad &\alpha_{j1} = \gamma - \epsilon_{123} - \epsilon_{145} + \delta_j - \delta_1 + 1, \quad \alpha_{l1} = \gamma - \epsilon_{123} - \epsilon_{145} + \delta_l - \delta_1 + 1,\\
&\alpha_{t1} = \gamma - \epsilon_{123} - \epsilon_{145} + \delta_t - \delta_1, \quad \alpha_{1i} = \gamma - \epsilon_{123} + \delta_1 - \delta_i + 1,\\
&\alpha_{ji} = \gamma - \epsilon_{123} + \delta_j - \delta_i + 1, \quad \alpha_{li} = \gamma - \epsilon_{123} + \delta_l - \delta_i,\\
&\alpha_{ti} = \gamma - \epsilon_{123} + \delta_t - \delta_i, \quad \alpha_{1k} = \gamma - \epsilon_{145} + \delta_1 - \delta_k + 1,\\
&\alpha_{jk} = \gamma - \epsilon_{123} + \delta_j - \delta_k, \quad \alpha_{lk} = \gamma - \epsilon_{145} + \delta_l - \delta_k + 1,\\
&\alpha_{tk} = \gamma - \epsilon_{145} + \delta_t - \delta_k, \quad \alpha_{1m} = \gamma + \delta_1 - \delta_m, \quad \alpha_{jm} = \gamma + \delta_j - \delta_m,\\
&\alpha_{lm} = \gamma + \delta_l - \delta_m, \quad \alpha_{tm} = \gamma + \delta_t - \delta_m.
\end{aligned}$$

4. *Bertini L-Curves with not more than 9 Multiple Points.* Since the P-curves of the planar Cremona transformations are Bertini L-curves [2; 4, p. 22], the algebraic expressions for r_i, s_j, α_{ji} in (4), (5) $\cdots$ (10) give expressions for the order t and multiplicity t_i of all the Bertini L-curves with not more than 9 multiple points. Each such Bertini curve occurs at least once, but may occur more often in this set of expressions.

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