The Cremona Plane

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1 Introduction – History

1.1 The geometry of Cremona is the birational geometry of the (complex) plane. It is based on the Cremona transformations or birational transformations of the projective plane introduced and studied by Luigi Cremona (°1820-†1903). Here birational means that the transformation and its inverse are rational. Let us recall what this means.

1.2 We choose an algebraically closed field k. Let $\mathbf{A}$ (respectively $\mathbf{P}$) be an affine (respectively projective) space over k equipped with a system of Cartesian (respectively homogeneous) coordinates. Let V be an algebraic variety over k, i.e., V is a subset of $\mathbf{A}$ (respectively $\mathbf{P}$) defined as the set of zeros of a system of polynomial equations (respectively homogeneous polynomial equations). Actually, we consider various such spaces $\mathbf{A}$, $\mathbf{A}'$, $\mathbf{P}$, $\mathbf{P}'$ and varieties $V \subseteq \mathbf{A}$, $V' \subseteq \mathbf{A}'$ (respectively $V \subseteq \mathbf{P}$, $V' \subseteq \mathbf{P}'$). We consider a *rational transformation* φ from V in V', i.e., $\varphi : V \dashrightarrow V' : x \mapsto x'$ such that the coordinates of x' are rational functions (respectively homogeneous polynomials) of the coordinates of x. Observe that in the projective case, we do not need quotients of polynomials since homogeneous coordinates are determined up to a multiple, which can be chosen as a common denominator. Remark that φ is a function rather than a mapping. It is not necessarily defined on all of V. It is defined on the complement of some proper subvariety. We indicate this by the notation $\varphi : V \cdots \to V'$. We call φ a *birational transformation* if there exists a rational transformation $\varphi' : V' \cdots \to V$ such that both $\varphi\varphi'$ and $\varphi'\varphi$ are the identity on their domain, which is again the complement of some proper subvariety. Up to the image of points of a proper subvariety, φ' is unique.

This provides an explanation of the Cremona transformations.

1.3 We give an example. Let V be the affine plane over k and define φ as $(x, y) \mapsto (x', y')$, with $x' = \frac{1}{x}$ and $y' = \frac{1}{y}$. This is not defined on the subvariety with equation $xy = 0$. Hence the inverse φ' is φ itself and $\varphi'\varphi$, $\varphi\varphi'$ have the same domain as φ. So φ is a birational transformation.

This birational transformation can be seen in the projective plane where it is defined by $(X, Y, Z) \mapsto (X', Y', Z')$, with $X' = YZ$, $Y' = ZX$, $Z' = XY$. This is called a *quadratic transformation* and we shall need this specific one several times afterwards (see e.g. the Theorem of Nœther below). It is undefined if two of the three coordinates are zero, i.e., in the points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. The inverse is $\varphi' = \varphi$. We see that $\varphi\varphi'$ is given by $(X, Y, Z) \mapsto XYZ(X, Y, Z)$, which is the identity in the complement of the subvariety with equation $XYZ = 0$. Observe that this domain is smaller than the domain of $\varphi = \varphi'$.

1.4 This is a good moment to say that "contemporary algebraic geometry" expresses the birational transformations in two ways.

In the first of these, due to André Weil, one works with indeterminates and "generic points". See A. WEIL, Foundations of algebraic geometry, *Amer. Math. Soc.*, New York, 1946. This view is not used very often.

In the second approach, the Zariski topology on $\mathbf{A}$ (respectively $\mathbf{P}$) and on V is used. A birational transformation $V \cdots \rightarrow V'$ is an isomorphism of an open dense subset of V onto an open dense subset of V'. Then Cremona transformations may be composed because the intersection of two open dense subsets is an open dense subset. In this way, the birational transformations of V constitute a group and this is the *Cremona group*. To see the connection with the previous paragraphs, let us just mention that in the Zariski topology the complements of proper subvarieties are precisely the open sets.

1.5 At this stage we may remember the viewpoint of Felix Klein and consider the *Cremona geometry* (or *Cremonian geometry*) as the geometry of properties invariant under the Cremona group. Let us remark here that for a long time algebraic geometry was a synonym of birational geometry. Then algebraic geometry was developing on its own. Recently, birational geometry has regained a more important role.

1.6 In the birth of Cremona geometry, Paul Libois had the major idea (at the end of the 1930s). Moreover, he was pushing his student Pierre Defrise towards the development of the birational geometry of the algebraic surfaces and this study extended to algebraic varieties. Defrise wrote his "thèse d'agrégation" (habilitationschrift) on this subject. We refer to

P. LIBOIS, La synthèse de la géométrie et de l'algèbre, CBRM, Colloque de géométrie algébrique, Liège, 1949, pp. 143 – 153 (Written in close collaboration with P. Defrise and J. Tits).

P. DEFRISE, Etude locale des correspondances rationnelles entre surfaces algébriques, *Mém. Soc. R. Sc. Liège*, 4è série, t.IX, 1949.

P. LIBOIS and P. DEFRISE, Sur les notions de point et de courbe en gémétrie birationnelle des surfaces, *C. R. Congrès Sc. Math. Liège*, 1939.

P. LIBOIS, Elaboration d'une axiomatique du plan de Cremona, *Accad. Nazio. Lincei*, 1973, 115 - 123.

In 1949, Libois asked me (= Jacques Tits) to develop a geometry of Cremona. I succeeded in writing a system of axioms for the Cremona plane in terms of a concept of Cremonian point, invariant under the Cremona group and so I solved the problem raised by Libois.

I wrote two manuscripts that have remained unpublished.

J. TITS, Caractérisation axiomatique de l'ensemble des points d'un plan crémonien, 1950.

J. TITS, Les courbes exceptionnelles, 1950.

For the time being, those manuscripts are lost. In the 1960s, a handwritten copy (with a few gaps) was made by Francis Buekenhout and this copy remains available.

The goal of the present lectures is to present the results of 1950 together with further thoughts and ideas.

1.7 P. Libois, P. Defrise and I were influenced by notions due to O. Zariski. After P. Libois, we refer to

O. ZARISKI, Polynomial ideals defined by infinitely near base points, *Amer. J. Math.* **60**, 1938.

O. ZARISKI, The reduction of the singularities of an algebraic surface, *Ann. of Math.*, v. 40, 1939.

O. ZARISKI, Local uniformization on algebraic varieties, *Ann. of Math.*, v. **41**, 1940.

O. ZARISKI, The compactness of the Riemann manifold of an abstract field of algebraic functions, *Bull. Amer. Math. Soc.*, v. **50**, 1941.

2 Cremonian Notions

2.1 As mentioned earlier, our study occurs outside of some subset that varies from one moment to the other. On what can one rely in these circumstances? A true socle for the rest of the lectures is the *field of rational functions* in two variables.

2.2 Let $\varphi : V \cdots \rightarrow V : (x,y) \mapsto (x',y')$ be any birational transformation. Here x' and y' are rational functions in x, y. Since φ is birational (and hence has an "inverse"), the fields $k(x,y)$ and $k(x',y')$ are equal. The map $(x,y) \mapsto (x',y')$ defines an automorphism of $k(x,y) = \mathbb{K}$, fixing all elements of k. This field is the primitive invariant of our Cremonian geometry. The Cremona group is the group of automorphisms of the field $\mathbb{K}$ fixing k pointwise. Such a statement can be made for any algebraic variety.

2.3 If we look for invariants of the Cremona group, it should be possible to express them in terms of $\mathbb{K}$. However, this is not a geometric viewpoint. We would like geometric notions. There will be a dialog between $\mathbb{K}$ and the geometry.

3 Models of $\mathbb{K}$

3.1 We shall elaborate a notion of Cremonian point on the basis of a preliminary notion of *model* of $\mathbb{K}$.

3.2 Consider the projective plane $\mathbf{P}$ over k with homogeneous coordinates (X, Y, Z). Here $\mathbb{K} = k(X/Z, Y/Z)$. The plane $\mathbf{P}$ is a model of $\mathbb{K}$ because we can choose coordinates such that $\mathbb{K} = k(X/Z, Y/Z)$. Of course, the field $\mathbb{K}$ admits (infinitely) many projective plane models — which are isomorphic, though. The field $\mathbb{K}$ has other models. For instance affine planes. These are incomplete models because affine planes are not varieties. There are still more models. We give a few examples.

(*i*) A quadratic cone in 3-dimensional projective space. We can write equations allowing to transform birationally the cone in a projective plane. We get the same field $\mathbb{K}$ of rational functions on the cone. This is a model with a singularity.

(*ii*) A non-degenerate quadric of Witt index 2 in a 3-dimensional projective or affine space.

(*iii*) A rational normal cubic scroll. In a 4-dimensional projective space, consider a line ℓ and a plane π that are disjoint, together with a conic Γ in π. Let there be given a projective correspondence between ℓ and Γ. The union of the lines joining corresponding points is a rational surface, called a rational normal cubic scroll. The field of rational functions on that surface is again isomorphic to $\mathbb{K}$. The surface is another model of $\mathbb{K}$.

1. Every cubic surface in a 3-dimensional projective space is another rational surface, a model of $\mathbb{K}$.

In summary, a model is not necessarily complete (e.g. an affine plane), it may be singular (e.e. a cone) and it may be complete as well.

3.3 In order to have a precise notion of model, we must add something to the above ideas. For k as earlier, in all models there arises a field $\mathbb{K} = k(x, y)$ of rational functions over k. This is a purely transcendental extension of k of degree 2. The Cremona plane should be a common abstraction of all models; therefore we need, on top of a geometric model, an *identification* of the geometric model with the field $\mathbb{K}$. So we define a *model* of $\mathbb{K}$ to be a rational surface V on k together with a given isomorphism (called *identification*) of $\mathbb{K}$ onto the field of rational functions on V.

Let V and V' be two models (there is a slight abuse of language here: in fact we have to include the identification in the notation). Since the fields of rational functions $k(V)$ and $k(V')$ are put into correspondence with $\mathbb{K}$ thanks to the given identifications, there arises a distinguished isomorphism of $k(V)$ onto $k(V')$.

3.4 We give an example. Let x, y be elements of $\mathbb{K}$ that constitute a transcendental basis of $\mathbb{K}$. They define an affine plane with "coordinates x, y" that we now call the *specter of* $k[x, y]$ and denote by Spec($k[x, y]$). The same model can have different systems of coordinates.

Consider x and $z = yx^{-1}$ that constitute another base of transcendence of the same field. They define an affine plane model with coordinates x, z. Consider A =Spec($k[x, y]$) and A' =Spec($k[x, z]$). They are two affine planes that are models of $\mathbb{K}$. There is a unique distinguished birational transformation from A to A' and from A' to A, provided by the structure of models, and given by the coordinate equalities $x = x$, $z = yx^{-1}$. These are not one-to-one but one-to-one off the lines $x = 0$ in A and $x = 0$ in A'.

The spirit of these lectures is that something occurs in the outside of the domain of definition of the birational transformation.

4 Cremonian Points

4.1 We are now ready to define Cremonian points. We start from $\mathbb{K}$ and from the class of all models of $\mathbb{K}$ (with their identifications). We consider all simple points of these models in order to define the Cremonian points. But we need an equivalence on those simple points. Let M and M' be two models and let φ be the distinguished birational transformation from M to M'. Then points $x \in M$ and $x' \in M'$ are called *equivalent* if x belongs to the domain of φ and if $\varphi(x) = x'$.

4.2 A *Cremonian point* is an equivalence class of simple points in all models of $\mathbb{K}$. We do not really want to work with such a definition but it is giving meaning to further developments.

4.3 If we consider one particular model M, we may think of a Cremonian point as either a point in M, or an equivalence class of pairs (x, φ) consisting of a point in another model M' and the distinguished birational transformation $\varphi : M' \cdots\to M$, where x is not in the domain of φ, and where two such pairs (x, φ) and $(x'\varphi')$ with $\varphi' : M'' \cdots\to M$, $x' \in M''$ are equivalent if there exists $\varphi'' : M' \cdots\to M''$ such that $\varphi''(x) = x'$ and $\varphi = \varphi' \circ \varphi''$ (remember that this equality means "everywhere except possibly on a proper subvariety")

So if we fix a model M, then the Cremonian points are the simple points p of M, together with "interpretations" of simple points p' of other models M' with respect to the distinguished birational transformations $M' \cdots \to M$ that are not defined in p'. We shall see examples of such "interpretations" soon, but first we ask ourselves how a Cremonian point can be seen in $\mathbb{K}$.

4.4 Without loss of generality, we may take an affine model with coordinates x and y, and we may consider the point $x = 0$, $y = 0$. We associate to the point the local ring of rational functions defined in that point, i.e., the ring $k[x, y] \subseteq \mathbb{K}$. Note that the maximal ideal of this ring is the set of all rational functions which vanish on the point $(0, 0)$ in question.

Now let us have a look at some interpretations of points in models which do not contain the point as a simple point. We start with an example. Consider the quadratic birational transformation $(X, Y, Z) \mapsto (YZ, ZX, XY)$ defined above. The point $(0, 0, 1)$ is not in the domain, since it gets formally mapped onto the triple $(0, 0, 0)$. But in fact, this triple $(0, 0, 0)$ is the result of the calculation $(0 \cdot 1, 0 \cdot 1, 0 \cdot 0)$, So, our feeling says that the last zero is more zero than the previous ones. We can see that as follows. Let $\ell \in k$ be arbitrary and consider the point $(X, \ell X, 1)$, $X \neq 0$. This is mapped onto the point $(\ell X, X, \ell X^2)$, which represents the same point as $(\ell, 1, X)$. Now putting $X = 0$ (or "letting X converge to 0"), we see that we obtain the point $(\ell, 1, 0)$. This is an arbitrary point on the line with equation $Z = 0$ (allowing $\ell = \infty$, which boils down to interchange the first and second coordinate and put ℓ equal to 0). Moreover, the point $(X, \ell X, 1)$ is in the direction ℓ with respect to $(0, 0)$ (i.e., the slope of the joining line is exactly ℓ), and this corresponds to the point $(\ell, 1, 0)$ on the line $Z = 0$. So, in conclusion, we may say that the point $(0, 0, 1)$ corresponds to the line $Z = 0$, and the different directions from $(0, 0, 1)$ (a direction can also be called a neighboring point) correspond to the points on that line. Another way of seeing this: a curve through $(0, 0, 1)$ for which the tangent line in $(0, 0, 1)$ has slope ℓ will be transformed into a curve through the point $(\ell, 1, 0)$.

In summary, the directions at a point p are new points, creations, that the Italian geometers were calling *infinitely neighboring points of* p. In the example above, the infinitely neighboring points of $(0, 0, 1)$ are mapped on ordinary points of the line $Z = 0$.

In the sequel, we shall call infinitely neighboring points just neighboring. The terminology of contemporary algebraic geometry would rather be to call them "nearby", but the Italians were using this term for another concept that we shall need as well.

If p is a point, a neighboring point of p has itself neighboring points that we consider as second order neighboring points of p. The transformation mapping a neighboring point on a point, maps a second order neighboring point on a (first order) neighboring point.

4.5 For example, let us have a look at the second order neighboring points of the point $(0, 0, 1)$, and represent them as second order curves through $(0, 0, 1)$. For example, we consider the quadratic curve containing the points $(X, \ell X + mX^2, 1)$, for fixed $\ell, m \in k$. Such a curve not only gives a direction (first neighborhood), but also a "curvature" (second neighborhood), expressed by the parameter m (and visible in the "second derivative"). The images of those points are the points $(\ell + mX, 1, \ell X + mX^2)$. Putting $X = 0$, we obtain our old friend $(\ell, 1, 0)$, corresponding with a first order neighboring point of $(0,0,1)$; for $X \neq 0$, we can write $(\ell + mX, 1, \ell X + mX^2) = (\ell, 1, 0) + (mX, 0, \ell X + mX^2)$. The latter can be considered as the point $(m, 0, \ell + mX)$, which corresponds, putting $X = 0$ again, to a first order neighboring point of $(\ell, 1, 0)$. Note that in the case $\ell \neq 0$, by varying m, we obtain all directions, except the direction of $Z = 0$ itself. On the other hand, if $\ell = 0$, then we only obtain $(m, 0, 0)$, which lies on $Z = 0$, and hence at the point $(0, 1, 0)$ we only have the direction of $Z = 0$. Similarly, at the point $(1, 0, 0)$, we only have the direction of $Z = 0$. In conclusion, a point, together with its first and second order neighboring points is mapped by a quadratic birational transformation onto a line, all its points, all neighboring points of these points (except for the one in the direction of the line itself), except for two points, for which we only take the first order neighboring point in the direction of the line.

So we consider a point as the set of all its neighboring points (of any order). "Most" points in an order n neighborhood can be detected with curves of degree n through the point ("most", because in the previous example, for instance, we missed the second order neighboring points "in the direction of $Z = 0$"). Letting n approach infinity, we obtain a *place* (this is a *branch of a curve*). So a place p can be seen as an infinite sequence $(p_0, p_1, p_2, p_3, \ldots)$, where p_i, $i \in \mathbb{N}$, is an order i neighboring point of p_0 (which is the *center* of p in the given model). In a birational transformation, some points p_i may "blow up"; however there is a smallest index j such that each p_k with $k \geq j$ is transformed in a point. The place itself is transformed in a place. Since a place is a Cremonian notion, we should be able to interpret it in the field $\mathbb{K}$. This is due to Zariski. For functions of one variable the notion of place appears already in Dedekind and in Weber. We show how we can attach to a place an "epimorphism" $\epsilon : \mathbb{K} \to k \cup \{\infty\}$, where we calculate with ∞ as follows:

(*) For all $r \in k$, we have $r + \infty = \infty = \infty + r$, $r - \infty = \infty = \infty - r$, $r/\infty = 0$, $\infty/r = \infty$, and for all $r \in k^\times$, we have $r \cdot \infty = \infty = \infty \cdot r$ and $r/0 = \infty$.

(**) $\infty + \infty$, $\infty - \infty$, $0/0$, $0 \cdot \infty$ and ∞/∞ are not defined.

Because of (**), our "epimorphism" should not preserve the operations which are not defined in the image.

Let $f = \frac{r}{s}$ be any rational function (element of $\mathbb{K}$). Let $i \in \mathbb{N}$ be minimal with the property that $(r(p_i), s(p_i)) \neq (0,0)$. Then we define $\epsilon(f) = \frac{r(p_i)}{s(p_i)}$, where $r/0 = \infty$ for $r \in k^\times$, according to rule (*) above.

4.6 One might ask how we calculate $r(p_i)/s(p_i)$ for $i > 0$, and why i in the previous paragraph exists. Suppose p_0 is a simple point in our model, say $\mathbf{P}$, and suppose p_1 is an order 1 neighboring point of p_0. Without loss of generality, we may take for p_0 the point with coordinates $(0,0,1)$, and p_1 is the neighboring point in the direction $(1, \ell, 0)$. If $r(p_0) = 0$, then in order to calculate $(r/s)(p_1)$, we apply a birational transformation φ which maps p_1 onto a simple point, and then we calculate $\varphi(r/s)(\varphi(p_1))$. In order to show that i exists, we prove the formal equality

$$\varphi(r/s)(\varphi(p_1)) = \frac{\nabla(r)(p_0).(1,\ell)}{\nabla(s)(p_0).(1,\ell)}$$

(where in the numerator and denominator we have the usual dot product, and where ∇ denotes the gradient). Indeed, we may put

$$r(X/Z, Y/Z) = \sum_{j=1}^{n} F_j(X,Y)Z^{-j},$$

with n some natural number and F_j some homogeneous polynomial of degree j, for $1 \leq j \leq n$, and similarly

$$s(X/Z, Y/Z) = \sum_{j=1}^{m} G_j(X,Y)Z^{-j},$$

with m some natural number and G_j some homogeneous polynomial of degree j, for $1 \leq j \leq m$. We apply the birational transformation $\varphi : \mathbf{P} \cdots \to \mathbf{P} : (X,Y,Z) \mapsto (YZ, ZX, XY)$. Put $F_1(X,Y) = aX + bY$ and $G_1(X,Y) = cX + dY$. We obtain, noting that $\varphi(p_1) = (\ell, 1, 0)$,

$$\begin{aligned}
[\varphi(r/s)(X/Z, Y/Z)]_{(X,Y,Z)=(\ell,1,0)} &= \left[\frac{\sum_{j=1}^{n} F_j(Y,X)Z^{j-1}(XY)^{-j}}{\sum_{j=1}^{m} F_j(Y,X)Z^{j-1}(XY)^{-j}}\right]_{(X,Y,Z)=(\ell,1,0)} \\
&= \frac{a+b\ell}{c+d\ell} \\
&= \frac{\frac{\partial r}{\partial(X/Z)}(0/1,0/1) + \ell\frac{\partial r}{\partial(Y/Z)}(0/1,0/1)}{\frac{\partial s}{\partial(X/Z)}(0/1,0/1) + \ell\frac{\partial s}{\partial(Y/Z)}(0/1,0/1)} \\
&= \frac{\nabla(r)(p_0).(1,\ell)}{\nabla(s)(p_0).(1,\ell)}.
\end{aligned}$$

Hence it is clear that at each step, one evaluates polynomials of smaller degree, and hence this process must end.

Note that the calculations above show that

$$\varphi(r/s)(\varphi(p_1)) = \lim_{X/Z \to 0} \frac{r(X/Z, \ell X/Z)}{s(X/Z, \ell X/Z)},$$

if the left hand side is not equal to $0/0$.

4.7 In summary, consider a place p consisting of successively neighboring points $p_1, p_2, \ldots$ and a rational function $f = r/s$, where r and s are polynomials. If $f(p_1)$ exists we are done. Otherwise $r(p_1) = s(p_1) = 0$. Then consider $f(p_2)$, etc. For some smallest index i, there is a well defined $f(p_i)$ in $k \cup \{\infty\}$. This is by definition the value of f at the given place. Libois had seen that this Zariski creation is the basic Cremonian notion. It is the origin of my work in 1950. However, something more substantial is needed.

4.8 We can now see any Cremonian point a as the set of places with center a. For instance, in $\mathbf{P}$, the point $(0,0,1)$ is the set of all places $\epsilon : \mathbb{K} \to k \cup \{\infty\}$ with the property that, if $(r(0,0), s(0,0)) \neq (0,0)$, for two polynomials $r, s \in k[X/Z, Y/Z]$, then $\epsilon(r/s) = \frac{r(0,0)}{s(0,0)}$. If, on the contrary, $(r(0,0), s(0,0)) = (0,0)$, then we must take a limit for X/Z and Y/Z approaching 0. However, this can be done in several ways, depending on the ratio Y/X. This corresponds exactly with the directions in $(0,0,1)$, in other words, with the neighboring points of $(0,0,1)$. But once we fix this ratio, we obtain a subset of the set of places with center $(0,0,1)$: it is the set of places with as center a point neighboring $(0,0,1)$. Hence, the neighboring points of a point p partition the set of places with center p into sets of places with center such neighboring points of p.

A *first order neighboring point* of a point p is by definition a point of the projective line $\mathcal{D}_p$ of directions in p in some model $\mathbf{A}$ where p is a simple point. We claim that this is independent of the model $\mathbf{A}$. Indeed, we may suppose that in some model $\mathbf{A}$ the point p corresponds with $x = 0, y = 0$, with x, y a basis of $\mathbb{K}$. The derivative $\frac{dy}{dx}$ provides coordinates in $\mathcal{D}_p$ with values in $k \cup \{\infty\}$. If x', y' is another basis of $\mathbb{K}$ such that $x' = 0$ and $y' = 0$ also defines p, then, by the well-known chain rule, we have

$$\frac{dy'}{dx'} = \frac{\left(\frac{\partial y'}{\partial x}\right)_{(0,0)} + \left(\frac{\partial y'}{\partial y}\right)_{(0,0)} \frac{dy}{dx}}{\left(\frac{\partial x'}{\partial x}\right)_{(0,0)} + \left(\frac{\partial x'}{\partial y}\right)_{(0,0)} \frac{dy}{dx}},$$

which clearly defines a projective transformation, hence the claim.

4.9 Once more, we consider the standard quadratic transformation $(X, Y, Z) \mapsto (YZ, ZX, XY)$. The point $(0,0,1)$ becomes a set of places on the one hand and, on the other hand, it seems to become the line $Z = 0$.

We therefore consider a line as the union of its points, hence as a set of places. Since we know how to compute transforms, we see that there are two special points, say a and b, on the line, that a and b are not in the image of $(0,0,1)$ except for their unique neighboring point in the direction of the line.

Thus, a full line is not a point. A point may be seen as a line minus two points (on the line) plus the neighboring points of the latter in the direction of the line.

If we started from the model consisting of a cubic surface, a line of that surface is indeed a point as can be seen from the representation of the surface on a projective plane equipped with six points.

4.10 Here are some other examples of points.

(*i*) A projective conic minus five points that are replaced each by the neighboring point in the direction of the tangent to the conic.

(*ii*) Take a line in a projective plane. Remove one point and replace it by the neighboring point in the direction of the line. Remove another point and replace it by a second order neighboring point.

4.11 We remark that there is a correspondence between neighboring points and "Puiseux series" in x, y (series with fractional exponents). This is not developed here.

5 Nearby points

5.1 Consider a Cremonian point p, its neighboring points of all orders and the places with center p. Are all such places "equivalent"? Are they in the same orbit for the stabilizer of p in the Cremonian group? Are all neighboring points of order n for p in the same orbit?

We consider the model consisting of the affine plane $\mathbf{A}$ and put $p = (0,0)$.

Here are the representations of some two neighboring points of order n:

$$y = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x$$

and

$$y = b_0x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_{n-1}x.$$

Note that a_0 or b_0 can be zero; but we have to write it otherwise we could confuse with neighboring points of smaller order.

There is a Cremonian transformation fixing p and mapping the first curve on the second, namely

$$(x,y) \mapsto (x, y - (a_0 - b_0)x^n - (a_1 - b_1)x^{n-1} - \cdots - (a_{n-1} - b_{n-1})x).$$

This does not entail all neighboring points of order n for p. Indeed, we have the enormous orbit we just described (but be aware of the fact that it contains neighboring points which cannot be represented in exactly the same way as above, namely, the ones that correspond with some coefficients equal to ∞; these have to be represented by more complicated equations where the degree in y can be bigger than 1), but there are points inaccessible (by the Cremonian group) from it. One such class is provided by the "nearby points" which we are now going to describe.

5.2 Consider the point p of the previous paragraph and a neighboring point p_1 of the first order. We can see this in **A** as a line through p. Among the neighboring points of the second order of p (and which are neighboring p_1), there is one point, say p_2, in the direction of that line. This is a point *nearby* p. In fact, the Cremonian stabilizer of p and p_1 fixes p_2 and fixes all further neighboring points $p_3, p_4, \ldots$ in the direction of that line. Any such point is by definition a *nearby point of* p.

5.3 This leads us to containment with multiplicity. Indeed, we shall now give an example which shows that a nearby point of p which is neighboring of the second order is in fact contained with multiplicity 2 in p.

Consider again the quadratic birational transformation φ of the projective plane given by $(X, Y, Z) \mapsto (YZ, ZX, XY)$. The point $(0,0,1)$ can be viewed, by applying φ, as the set of points of the line $Z = 0$, except for the points $(1,0,0)$ and $(0,1,0)$, which we replace by their first order neighboring points "in the direction of $Z = 0$". Take the point $(1,1,0)$, and consider its first order neighboring point p "in the direction of $Z = 0$". This is a nearby point as we defined it in the previous paragraph, and, as a simple point, it is contained in $(0,0,1)$, but as a direction, it is contained in $Z = 0$. Hence, we can say it is contained two times in $(0,0,1)$. That motivates us to say that it is contained in $(0,0,1)$ *with multiplicity* 2. We will see other examples in the next section. It will be crucial to see unions of points as "sums" of points, where the sum refers to the sum of multiplicities. An algebra of points is arising. This leads us to the notion of a "figure". But before defining this, we mention one property of containment, which will be helpful in the setting of "sums".

5.4 **Proposition 0.** *Consider a point p as a set of places. Then those points contained in p are precisely the neighboring points of p. The set of points has the structure of a tree.*

The proof is not given here. If q is a point contained in p, one shows that there is a unique sequence $p = p_0, p_1, \ldots, p_n = q$ in which any two consecutive points are neighboring of the first order. Not all such sequences are equivalent under the Cremona group, as there can arise nearby points. This matter leads to Enriques schemes, which we shall explain in more detail in 7.3.

6 Figures

6.1 The points will be primitive elements in our axiomatization. The other primitive elements will be the "figures". It will turn out that points are special cases of figures. In principle, a figure is a union of points. But there are two ways to look at unions. Either set-theoretically (and we obtain "flat figures") or algebraically (with multiplicities). It will be crucial that we consider the second way, and the axiomatization we aim at is based on it.

6.2 A *flat figure* is the set-theoretic union of Cremonian points (which, on their turn, are sets of places). So we can take intersections and unions of flat figures, or of single points, etc. But in the previous section, we saw that there is a notion of multiplicity, and an algebra of points based on it. We define a *figure* $\mathcal{F}$ (with multiplicities) as a map from the set of points of a flat figure to the set of natural numbers $\mathbb{N}$, which assigns to a point p the multiplicity n with which it is contained in the figure. We denote $[\mathcal{F} : p] = n$ for the multiplicity of containment of p in $\mathcal{F}$. The figure is called *effective* if $[\mathcal{F} : p] \geq 0$, for all points p. Instead of unions of figures, we take the sum of figures with multiplicity; instead of intersections, we take lower bounds.

6.3 Of course, not every figure is "interesting". There are two classes of interesting figures: complete algebraic curves in a projective plane model, and the neighborhoods of points in the different models.

In fact there are formulae to calculate the multiplicity of points in a curve, or in another point, but we will not give the precise expressions. Instead, we will give some geometric feeling for the notion of multiplicity.

First we take the example of a curve $\mathcal{C}$ and a point p in a certain model M. As we know, the Cremonian point p can be an ordinary point in M. In that case, $[\mathcal{C} : p]$ is easy to calculate. In an affine plane model, and with p the origin $(0,0)$, it is just the order of the polynomial defining $\mathcal{C}$. Now we ask ourselves: what does $\mathcal{C}$ look like in other models?

We remark that the multiplicity can be defined in such a way that the notion of finite linear combination with integer coefficients is invariant under the Cremona group. With integer coefficients, we mean that, formally, a point can also be contained in a figure with multiplicity -1, for instance. Now, the Theorem of Nœther says that the Cremona group is generated by the projective transformations and one quadratic birational transformation (which we can choose to be our old friend $(X, Y, Z) \mapsto (YZ, ZX, XY)$ as above). The first complete proof of this theorem is due to Castelnuovo (for $\mathbb{C}$), and to Nagata (for arbitrary algebraically closed fields). Let us also remark that there is a similar theorem for general fields.

To see the multiplicities geometrically, it helps if we know about the procedure of Hopf. This is commonly known as "blowing up" and "blowing down". What one does, is replace a point by a line. For instance, if we replace in a projective plane a point by a line, then we obtain the normal rational cubic scroll (see above). Indeed, we perform the transformation $(X, Y, Z) \mapsto (X^2, XY, Y^2, XZ, YZ)$ from the projective plane into projective 4-space. If the coordinates in the 4-space are $(X_1, X_2, X_3, X_4, X_5)$, we see that the point $X = Y = 0$ becomes the line $X_1 = X_2 = X_3 = 0$. The notion of a line depends on the model we consider. A line of the projective plane is larger than the above. The lines of a projective plane have "self-intersection" equal to 1 while the above has a "self-intersection" equal to -1. A Cremonian point has also self-intersection -1. The advantage of such a blowing up is that it takes place at one point, but the disadvantage is that it transforms the model.

Compare this with our standard quadratic birational transformation: it does not really transform our model, but it blows up three points at once, and at the same time blows down three lines!

Let C be a curve and consider a (simple) point p on C. Let p' be the first order neighboring point of p in the direction of the curve C. If we blow up at p, then the curve C transforms into the sum of a curve C' and a line L (this line is the transform of the point p), and the intersection point is the transform of the point p': it is contained two times in the transform of C: once in the curve C', once in the line L.

Similarly, we consider a point p, and view it as a line L with two points replaced by its first order neighboring points in the direction of L. We blow this up at a point q of L distinct from the two special points and see that the first order neighboring point of q in the direction of L is contained two times in p. This is a nearby point of p.

In the spirit of flat figures, a point was the union of all its neighboring points. But now we can say that a point is the sum of all its nearby points. It follows that the multiplicity of a point p in a point q (in which it is contained) is the sum of multiplicities in q of all points nearby p.

We can now say that the transform of the point $X = 0$, $Y = 0$ under the standard quadratic birational transformation is a line minus two points plus the two neighboring points in the direction of the line (namely, the two nearby points). So, in fact, a line is the sum of three points.

7 Some properties

Before stating the axioms, we mention some properties of Cremonian points and containment of points in each other.

Proposition 1. *Given two Cremonian points A and B, there are only a finite number of points which contain A and are at the same time contained in B.*

In the situation of the proposition we denote $B > A$. If $n + 1$ is the number of points referred to in that proposition, then we say that A is a neighbor of B *of order* n.

7.2 **Proposition 2.** *Let A be a Cremonian point and let A_1 be a neighboring point of A of order 1. Then there exists a sequence of points $A_1 > A_2 > A_3 > \cdots$ neighbors of A of order $1, 2, 3, \ldots$ such that one (and hence every one) of the following holds.*

(*i*) $[A : A_i] = i$.

(*ii*) *For every point X, we have* $[A : X] = \sum_i [A_i : X]$.

(*iii*) *In a projective plane model where A is a line L minus two special points and A_1 is a point of that line (but not one of the two special points), the point A_i is "in the direction" of L.*

The latter means that, if we blow up $A_1, A_2, \ldots, A_{i-1}$, then the intersection of A_{i-1} with the rest of A is exactly A_i.

(*iv*) *The statement under (iii) is true in every model.*

The points $A_1, A_2, \ldots$ are the nearby points of A in the direction of A_1. Considered as a figure, A is the sum of all its nearby points. One talks about a *total curve = pure curve + nearby points.*

7.3 One can now define the so-called *Enriques schemes.* Given two points A and B with $A > B$, we know there are a finite number $n+1$ of points "between" A and B (by Proposition 1). We picture these on a path from A to B. Suppose that C_i, $0 < i \leq n$ is the neighbor of order i of A containing B. The edges (C_{i-1}, C_i) and (C_i, C_{i+1}) are drawn perpendicular if C_{i+1} is nearby C_{i-1}. Moreover, if C_{i+k}, $k > 0$, is still nearby C_{i-1}, then the segment (C_i, C_{i+k}) is a straight line. In all other cases, the angle at C_i of the segment (C_{i-1}, C_{i+1}) is "very obtuse", but not equal to 180 degrees. One can now calculate $[A : B]$ by recurrence using Proposition 2 (*ii*) above (it is equal to $[C_1 : B]$ if (A, C_1) is not perpendicular to (C_1, C_2), otherwise it is equal to the sum of all $[C_i : B]$ for all $i \in \{1, 2, \ldots, n\}$ such that the segment (C_1, C_i) is straight). For instance, if B is nearby A, then (A, C_1) is perpendicular to (C_1, C_2); all $[C_i : B]$ are equal to 1 and $[A : B]$ is the sum of all those numbers, hence it is equal to n, which is in conformity with Proposition 2 (*i*).

8 The exceptional curves

8.1 An exceptional curve is a rational total curve in a non-singular model. Its points are the ordinary points of the model lying on the curve. If we delete a point of the curve, then we say that we *thin* the curve. If we add a *transversal point* to the curve (i.e., a point which, as a curve in another model meets the original curve in a point which is ordinary for each of the two curves) then we say that we *thicken* the curve.

Proposition 3. *By thinning and thickening an exceptional curve we get an exceptional curve.*

Proposition 4. *There exists an invariant called the* degree *of the exceptional curve which raises by one unit each time we thicken the curve. A thinning diminishes the degree by one unit.*

The exceptional curves of degree -1 are the points. A projective line is an exceptional curve of degree 1 (this follows from the fact that a line minus two points is a point). One other example: a conic is an exceptional curve of degree 4. Hence a conic minus five points is a point.

Proposition 5. *A Cremonian transformation preserves the degree of any exceptional curve.*

We end by noting that each point P of an exceptional curve C of degree 0 is ordinary, i.e., $[C : P] = 1$.

8.2 Given an exceptional curve, it is suitable to produce a model in which it is a genuine curve.

Consider projective spaces of respective dimensions m and n disjointly embedded in a projective space of dimension $m+n+1$. In each component, give an irreducible normal rational curve. Establish a projectivity between the two curves. Consider all lines joining two corresponding points. The union of these lines is a ruled rational surface, a model of $\mathbb{K}$. The normal curves are exceptional curves of degree $n-m$ and $m-n$, respectively.

A particular case is a cone projecting a normal curve. Its singular point is an exceptional curve of negative degree.

An other example: on a ruled quadric of a projective 3-space, each line of the quadric is an exceptional curve of degree 0. This is the particular case where $m = n = 1$.

8.3 For an exceptional curve of degree other than 0, the ordinary points do not depend on the model. If C is an exceptional curve of degree 0, a rather curious phenomenon occurs. Every point contained in C with multiplicity 1 in any model is an ordinary point. Hence, such a curve has an enormous amount of ordinary points. The proof requires de Jonquières transformations.

If we remove a point from an exceptional curve of degree 0, we get a point. Such a curve has sometimes been called a *half-line*. A de Jonquières transformation can be defined as follows in a plane model with coordinates (X, Y). Map (X, Y) on (X', Y') with $X' = X$ and $Y' = Y + R(X)$, where R is a rational function of X.

9 Axiomatization

We are now ready to state 14 axioms which characterize the Cremonian geometry over an algebraically closed field. No effort was made in order to reduce the axiomatic.

Let there be given a set $\mathcal{F}$ of *figures*, which is an ordered (additive) group without torsion. Let $\mathcal{P}$ be a subset of the set of positive figures, the elements of which we call *points*. We require that $\mathcal{P}$ generates $\mathcal{F}$. We define the *containment* $[A : B]$ (for two points A, B)

as the maximum number n for which $A - nB \geq 0$. Further, we say that two figures F_1 and F_2 are *disjoint*, denoted by $F_1 \cap F_2 = 0$, if no point is contained in both F_1 and F_2 (meaning if for no point P we have $P < F_1$ and $P < F_2$).

Ax 1 *If $A, B \in \mathcal{P}$, then $[A : B] < \infty$.*

Ax 2 *If $\{A_i | i \in I\}$ is a finite set of positive figures, and if a_i is an arbitrary integer for each $i \in I$, then $\sum a_i A_i \geq 0$ if and only if for every point P, there exists a point $Q \leq P$ such that $\sum a_i [A_i : Q] \geq 0$.*

Ax 3 *For every $P, Q \in \mathcal{P}$ with $P \not\leq Q$, there exists $R \in \mathcal{P}$ such that $R \leq Q$ and $R \cap P = 0$.*

Ax 4 *For every two points P, Q contained in any point R, we have either $P \cap Q = 0$, or $P \leq Q$, or $Q \leq P$.*

Ax 5 *If $P \leq Q$ for two points P, Q, then there are only a finite number of points R with $P \leq R \leq Q$.*

From these five axioms, one deduces that the points in Ax 5 are totally ordered. If there are $n + 1$ such points, then we say that P is a neighbor of Q of *order n*. Let $Q \geq P_1 \geq P_2 \cdots \geq P_n = P$ be these points, then we say that P is *nearby* Q if $[Q : P_i] = i$, for all i.

Ax 6 *If P is a point nearby the point Q, then there exists a unique point nearby both P and Q.*

Ax 7 *For every point Q and every point X, we have that $[Q : X]$ is equal to the sum of all $[Q' : X]$, where Q' ranges over the set of points nearby Q.*

Ax 8 *Any point has at least three distinct neighbors of order 1.*

Concerning the latter axiom, it would be interesting to see what happens if we replace "at least three" by "exactly 2". Do we obtain a "thin Cremonian plane", playing the role of an *apartment* in the sense of *buildings*? The conjecture is that there is a unique thin Cremonian plane.

Two points are called *transversal* if they share a neighboring point of order 1 (called the *intersection point* below) and nothing more. A *half line* is the sum of two transversal points. Two half lines D, D' are called *transversal* if there exists a point P such that $[D : P] = [D' : P] = 1$ and $(D - P) \cap (D' - P) = 0$.

Ax 9 *If for a point P and a half line D we have $P \leq D$ and $[D : P] = 1$, then $D - P$ is a point transversal to P.*

Two disjoint half lines are called *parallel*, notation: $D \| D'$. For the sake of convenience we also call a half line parallel to itself; hence $D \| D$.

Ax 10 *Parallelism is a transitive relation. i.e.,* $D \| D'$ *and* $D' \| D''$ *imply* $D \| D''$.

Ax 11 *If* $D \| D'$*, then every half line transversal to* D *is also transversal to* D'.

Ax 12 (The axiom of Euclid) *Let* D *be a half line and* P *a point disjoint from* D*. Then there exists a unique half line* D' *containing* P *and parallel to* D.

A parallel class of half lines will be called a *bundle*. Two bundles are *perspective* if there exists a common transversal half line with same family of intersection points. "Equality of intersection point" defines a bijection between the bundles, which we call a *perspectivity*. A product of perspectivities is called a *projectivity*. One can show that the group of projectivities of a bundle onto itself is triply transitive. The next axiom says that it is sharply 3-transitive.

Ax 13 *Every projectivity of a bundle onto itself fixing at least three elements is the identity.*

Ax 14 *Every subset of* $\mathcal{P}$*, closed under the relation of inclusion, which satisfies the axioms above coincides with* $\mathcal{P}$.

We can now state the main result of these lectures.

Theorem. *Under the condition above, the group of projectivities of a bundle onto itself is the projective group* $PSL_2(k)$ *of an algebraically closed field* k*. Moreover* $\mathcal{P}$ *is isomorphic to the set of Cremonian points defined earlier from the field* $k(x, y)$*. In particular, the group of automorphisms of* $(\mathcal{F}, \mathcal{P}, \leq)$ *is isomorphic to the automorphism group of* $k(x, y)$ *leaving* k *invariant.*

The proof is long but not very difficult. One of the ideas is to define the de Jonquières transformations.