ON PLANE CREMONA TRIADIC CHARACTERISTICS.

By Charles C. Torrance.

The problem of finding plane Cremona transformations having only three groups of basis points, that is, transformations with triadic characteristics, has been attacked by Ruffini,[1] Montesano,[2] and Farnum.[3] In this paper twelve infinite sequences of two-parameter triadics are obtained, all of which have for one of their groups a single point of highest multiplicity. These triadics result directly from the development of a new procedure [4,5,6] whereby all geometric characteristics may be found. The principal feature of this procedure is its perfect regularity and simple explicit formulation.

1. It is a well-known theorem that if P, Q, and R are basis points of multiplicities $p \geqq 0$, $q \geqq 0$, and $r \geqq 0$ of a plane Cremona transformation T_n of order n and also are the basis points of a quadratic inversion T_2 with which T_n is compounded, the resultant transformation $T_{n'}$ of order n' has the same number of basis points at T_n of the same multiplicities except that it loses the points P, Q, and R, and gains three basis points P', Q', and R' of multiplicities $p' = p + e$, $q' = q + e$, and $r' = r + e$, while $n' = n + e$, where $e = n - (p + q + r)$. It will be convenient to regard the three points gained by $T_{n'}$ as the transforms of the three points lost by T_n.

2. We give the symbol P a particular connotation. If a transformation is to be successively compounded with T_2's and one of its basis points is called P, it is intended that P and each of its transforms are to be successively used as common basis points. If T_n is compounded with a T_2 having P, Q_1, and

1 "Sulla risoluzione delle trasformazioni Cremoniane," *Memorie Istituto Bologna*, Series 3, Vol. 8 (1877), pp. 457-525.

2 "I gruppi Cremoniani di numeri," *Atti Accademia Napoli*, Series 2, Vol. 15 (1914), No. 7.

3 "On triadic Cremona nets of plane curves," *American Journal of Mathematics*, Vol. 50 (1928), p. 357.

4 Montesano, "Su le rete omaloidiche di curve," *Rendiconti Accademia Napoli*, Series 3, Vol. 11 (1905), pp. 259-303.

5 Montesano, "Su i quadri caratteristici delle corrispondenze birazionali piane," *Rendiconti Accademia Napoli*, Series 3, Vol. 21 (1915), pp. 30-38, 69-79, 113-119.

6 Montesano, "Su alcuni problemi fondamentali nella teoria delle corrispondenze Cremoniane," *Rendiconti Accademia Napoli*, Series 3, Vol. 34 (1928), pp. 42-50, 89-97, 123-136.

R_1 as common basis points, $e = (n - p) - (q_1 + r_1)$, $n' = n + e$, $p' = p + e$, and $n' - p' = n - p$. If the resultant transformation $T_{n'}$ is then compounded with a T_2 having P', Q_2, and R_2 as common basis points, $e' = (n' - p') - (q_2 + r_2) = (n - p) - (q_2 + r_2)$. A sufficient condition that $e' = e$ is that $q_2 = q_1$ and $r_2 = r_1$. Hence, in a succession of compoundings involving a point P and its transforms, $(n - p)$ is invariant, and e is invariant if Q and R are each repeatedly taken with invariant multiplicity.

3. The characteristic (1) is known to be geometric. Let P be the single point of its first group. Compound it with a T_2 having its other two basis points general [i. e., of multiplicity zero in (1)]. The resultant is (2) since $e = x_{01}$. If this compounding is repeated, (2) is altered in the same way as was (1). The resultant of x_{11} of these compoundings is (3). In general, compounding with T_2's each having one basis point on P and two general will be referred to as compounding in the first way.

$$
\begin{array}{lllll}
(1)\ n=[x_{01}+1]; & 1^1, & 1^{x_{01}}, & & (2x_{01}-1)^1. \\
(2)\ n=[x_{01}+1+x_{01}]; & 1^{1+x_{01}}, & (1+2)^{x_{01}}, & & (2x_{01}-1)^1. \\
(3)\ n=[x_{01}+1+x_{01}x_{11}]; & 1^{1+x_{01}x_{11}}, & (1+2x_{11})^{x_{01}}, & & (2x_{01}-1)^1. \\
(4)\ n=[x_{01}+1+x_{01}x_{11}-1]; & 1^{1+x_{01}x_{11}-1}, & (1+2x_{11}-1)^{x_{01}}, & 1^{x_{01}-1}, & (2x_{01}-1-1)^1. \\
(5)\ n=[x_{01}+1+x_{01}x_{11}-x_{12}]; & 1^{1+x_{01}x_{11}-x_{12}}, & (1+2x_{11}-x_{12})^{x_{01}}, & {x_{12}}^{x_{01}-1}, & (2x_{01}-1-x_{12})^1.
\end{array}
$$

4. Compound (3) with a T_2 having one basis point on the last transform of P, one on a point of the second group, and one on a point of the third. The resultant is (4) since $e = -1$. If this compounding is repeated, (4) is altered in the same way as was (3). The resultant of x_{12} of these compoundings is (5). In general, compounding with T_2's each having one basis point on P, a second on a point of the second group, and the third on a point of any other group will be referred to as compounding in the second way.

5. The procedure for obtaining all geometric characteristics may be described in general terms as follows: suppose the characteristic

$$
(6) \qquad n; \quad 1^{z_{k1}}, \quad (1 + y_{k2})^{z_{k2}}, \quad y_{k3}^{z_{k3}}, \cdot\cdot\cdot, y_{ki}^{z_{ki}}, \cdot\cdot\cdot, y_{k2^{k+1}}^{z_{k2^{k+1}}}
$$

to be geometric, with $n = z_{k1} + z_{k2}$, where the y's and z's are polynomials in x_{rs}, $r = 0, 1, \cdot\cdot\cdot, k$, and $s = 1, 2, \cdot\cdot\cdot, 2^r$. An illustration of such a characteristic is (5). Take for P a point of multiplicity z_{k2} (involving a change of P) and write (6) in the form (7) so that P constitutes the first group.

$$
(7) \qquad n; \quad 1^{z_{k2}}, \quad 1^{z_{k1}}, y_{k2}^{z_{k2}}, y_{k3}^{z_{k3}}, \cdot\cdot\cdot, y_{ki}^{z_{ki}}, \cdot\cdot\cdot, y_{k2^{k+1}}^{z_{k2^{k+1}}}.
$$

$$(8) \qquad n' = [n + z_{k1}x_{k+1,1}]; \quad 1^{z_{k2}+z_{k1}x_{k+1,1}}, \quad (1 + 2x_{k+1,1})^{z_{k1}}.$$
$$y_{k2}^{z_{k2}}, \cdots, y_{ki}^{z_{ki}}, \cdots, y_{k2^{k+1}}^{z_{k2^{k+1}}}$$

$$(9) \qquad n' = [n + z_{k1}x_{k+1,1} - \sum_{i=2}^{2^{k+1}} z_{ki}x_{k+1,i}]; \quad 1^{z_{k2}+z_{k1}x_{k+1,1} - \sum_{i=2}^{2^{k+1}} z_{ki}x_{k+1,i}},$$
$$(1 + 2x_{k+1,1} - \sum_{i=2}^{2^{k+1}} x_{k+1,i})^{z_{k1}}, \; x_{k+1,2^{k+1}}^{z_{k1}-z_{k2^{k+1}}}, \cdots, x_{k+1,i}^{z_{k1}-z_{ki}}, \cdots,$$
$$x_{k+1,2}^{z_{k1}-z_{k2}}, \; (y_{k2} - x_{k+1,2})^{z_{k2}}, \cdots, (y_{ki} - x_{k+1,i})^{z_{ki}}, \cdots,$$
$$(y_{k2^{k+1}} - x_{k+1,2^{k+1}})^{z_{k2^{k+1}}}.$$

Compound (7) in the first way $x_{k+1,1}$ times, where $e = z_{k1}$, getting (8). Compound (8) in the second way $x_{k+1,i}$ times with the third basis point of the compounding T_2 on a point of multiplicity z_{ki}, i having successively all integer values from 2 to 2^{k+1}, getting (9), since $e = -z_{ki}$. As (9) is of the same form as (6) the procedure is inductive. The set of compoundings used to obtain (9) from (6) will be referred to as one step of the inductive procedure, and (6) itself will be referred to as the general characteristic resulting from k steps of this procedure. It should be noted that each general characteristic includes all preceding ones.

The only condition on the x's is that all the base numbers (i. e., the expressions giving the number of points in each group) must be non-negative, and if this condition is satisfied all characteristics obtained by this procedure are geometric, since (1) is geometric.

6. Theorem. *Every plane Cremona geometric characteristic can be obtained as the result of sufficiently many steps of the above procedure* [*starting from* (1)] *and proper evaluation of the x's.*

This theorem will be proved by showing that the inverse of the procedure will reduce a given characteristic to a particular case of (1). Let the characteristic of a given transformation T be n; y_h^h, $y_{h-1}^{h-1}, \cdots, y_1^1$, where h denotes the highest multiplicity in T. (In the course of this proof h will also denote the highest multiplicity in any particular transform of T.) Take one of the points of multiplicity h as P, and let the multiplicity of this particular point be denoted by p. The reason for the notation is that $(n-p)$ is always invariant, but $(n-h)$ is not.

7. If there exists in T a point Q (other than P) of multiplicity q, $n - p > q > (n-p)/2$, compound T with a T_2 having one basis point on P, one on Q, and one general. In the notation of paragraph 1, $e > 0$, $p' \geqq q' = n - p$, and $r' < (n-p)/2$, so that this compounding is in the

inverse of the second way. For simplicity we drop the primes in considering this and succeeding transforms of T. In this way all points Q may be removed, so that there remain only P of multiplicity $p = h$, points of multiplicity $(n - p)$, and points of multiplicity not greater than $(n - p)/2$. Before making further compoundings we use

8. Noether's Extended Theorem. In any geometric characteristic the sum of the first, third, and fifth highest multiplicities is greater than the order. It follows that there are at least three points of multiplicity greater than $(n - h)/2$. Hence, if $p = h$ and there is no point Q, there are at least two points (other than P in the event that $p = n - p$) of multiplicity $(n - p)$, and $p \geqq n - p$.

9. Compound T in its present form with a T_2 having one basis point on P and two on points A and B of multiplicity $(n - p)$. In obvious notation $e = -(n - p)$ and $a' = b' = 0$, so that this compounding is in the inverse of the first way. Let this compounding be repeated as many times as possible, that is, until $p < n - p$.

If now y_{n-p} is positive, $n - p = h > p = n - h$, and the group consists of but a single point. Since $h + p = n$, p is the second highest multiplicity, and T has been reduced to a transformation T' of the form (6) with P of the proper multiplicity, so that, if the procedure produces T' it also produces T.

10. If, however, y_{n-p} is zero, then $p < h \leqq (n - p)/2$, since T now contains no point Q. Compound T in its present form with a T_2 having its basis points on P, a point H of multiplicity h, and a general point K. Since $h' = n - p > p' \geqq k' = e \geqq (n - p)/2$ this compounding is in the inverse of the second way. But P is now a point of second highest multiplicity and $y_{n-p} = 1$. Hence in this case also T has been reduced to a transformation T' of the form (6).

11. As this last compounding is the inverse of the second way its use here is unwarranted. But the difficulty may be overcome without affecting T'. Because of the invariance of $(n - p)$ the effect on T of each compounding is independent of the others, so that the order of the compoundings may be changed without altering T' and a compounding may be omitted without impairing the effect on T of any of the others. Let K' be the point of multiplicity k' involved in the last compounding. If K' was present in the original T as a point Q, the continued product of the compounding which removed it, the last compounding of paragraph 9, and the compounding of

paragraph 10, is the identity. Hence they may all be omitted without affecting T'. If K' was not present as a point Q, the compounding introducing it is necessary, but may be performed in paragraph 7.

12. Now take the single point of multiplicity $(n-p)$ as a new P and repeat the above process, noting that the new value of $(n-p)$ is less than the old one. By sufficiently many repetitions of this process it is possible to reduce the value of $(n-p)$ to one, and when this has been done the result must necessarily be a particular case of (1).[7]

13. This proof was conducted in such a way as to show that any characteristic may be obtained with P a point of highest multiplicity, not only in it, but throughout the process of getting it. Hence, in constructing T from (1) by the inverse of the above reduction, all x_{r1}, $r=0, 1, 2, \cdots, k$ (where k is the required number of steps in the inductive procedure) are positive. The vanishing of an x_{r1} by no means invalidates the procedure; it leads only to duplicates. We shall assume in the sequel that all such x's are positive.

14. If we now impose the condition that all but three of the base numbers vanish in the general characteristic resulting from k steps of the inductive procedure, $k>1$, it results that all the x's can be evaluated by six different sets of formulas in terms of x_{01} and x_{11} taken as independent parameters and in no other way. If, for successive values of k, these values of the x's are substituted in the general characteristics, triadics 1, 2, 3, 4, 5, and 6 result. (In these triadics we have written merely x_0 for x_{01} and x_1 for x_{11}.) We omit the proofs of these statements because of typographical difficulties.

Other formulas for evaluating the x's so that triadics result may be obtained by making two or more of the multiplicities equal in the general characteristics. For example, the condition $x_{01}=2$ makes every other pair of multiplicities equal, so that four, or even five base numbers may be positive without involving more than three different multiplicities. However, this particular condition leads only to special cases of the triadics given below, together with some half-dozen isolated triadics.

15. Let $\cdots, u_{-2}, u_{-1}, u_0, u_1, u_2, \cdots$ be the two-way sequence of polynomials satisfying the relations

$$(A_e)\qquad u_{i+1}+u_{i-1}=x_1 u_i,\ i \text{ even},$$
$$(A_o)\qquad u_{i+1}+u_{i-1}=\ \ u_i,\ i \text{ odd},$$

and the condition that

$$u_0=0,\qquad u_1=1.$$

7 Hudson, *Cremona Transformations*, p. 98.

It follows immediately that all the u's are polynomials in the one variable x_1 and that $u_i = -u_{-i}$, since $u_{-1} = -1$. These polynomials are connected by many relations, among which the following are used in the sequel:

$$(B_e) \qquad u_{i\pm2} + x_1 u_i - 2u_{i\pm1} = u_{i\mp2}, \quad i \text{ even, signs dependent.}$$

$$(B_o) \qquad u_{i\pm2} + x_1 u_i - 2x_1 u_{i\pm1} = u_{i\mp2}, \quad i \text{ odd, signs dependent.}$$

$$(C1) \qquad u_{i+(j+1)} u_{i-(j+1)} = x_1 u_{i+j} u_{i-j} - u_{2j+1}, \quad (i \pm j) \text{ even.}$$

$$(C2) \qquad x_1 u_{i+(j+1)} u_{i-(j+1)} = u_{i+j} u_{i-j} - u_{2j+1}, \quad (i \pm j) \text{ odd.}$$

$$(C3) \qquad u_{i+(j+1)} u_{i-(j+1)-1} = u_{i+j} u_{i-j-1} - u_{2j+2}, \quad \text{all } (i \pm j).$$

$$(C4) \qquad u_{i+(j+2)} u_{i-(j+2)} = u_{i+j} u_{i-j} - u_{2j+2}, \quad (i \pm j) \text{ even.}$$

$$(C5) \qquad u_{i+(j+2)} u_{i-(j+2)} = u_{i+j} u_{i-j} - x_1 u_{2j+2}, \quad (i \pm j) \text{ odd.}$$

$$(C6) \qquad u_{i+(j+2)} u_{i-(j+2)-1} = u_{i+j} u_{i-j-1} - u_{2j+3}, \quad \text{all } (i \pm j).$$

Relations (B) are proven directly by substitution from relations (A). Relation $(C1)$ is proven by induction first for the case $j = 0$. Then relations $(C2)$ and $(C3)$ are proven directly for the case $j = 0$ by substitution from relations (A) and $(C1)$. Finally relations $(C1)$, $(C2)$, and $(C3)$ are proven for general j by showing that if they all hold for all proper i when $j = s$, then they all hold for all proper i when $j = s + 1$. Relations $(C4)$, $(C5)$, and $(C6)$ are proven directly by substitution from relations (A) and the other relations (C). Details of the proofs are omitted because they offer no difficulty.

As a digression we may point out an interesting property of these u-polynomials. Transform u_i into u'_i by the substitution $x_1 = x'_1 + 2$. The roots of the equation $u'_i = 0$, where i is positive and even, are equal to $2 \cos (2n\pi)/i$, $n = 1, 2, \cdots, (i-2)/2$.

16. The following triadics can be shown to be geometric and their conjugates can be determined by Montesano's method [6]:

(1)	$n=[x_1b_ka_k+x_1+1]$;	$1^{(n-1)-x_1u_ka_k}$,	$(2x_0-1)^{x_1u_ka_k}$,	$(2x_1)^{u_{k+1}a_k+1}$,	k even.
	$n=[\ b_ka_k+x_1+1]$;	$1^{(n-1)-\ u_ka_k}$	$(2x_0-1)^{\ u_ka_k}$,	$(2x_1)^{u_{k+1}a_k+1}$,	k odd.
(1c)	$n=[x_1a_kb_k+x_1+1]$;	$1^{(n-1)-x_1u_kb_k}$,	$(2x_0-1)^{x_1u_kb_k}$,	$(2x_1)^{u_{k-1}b_k+1}$,	k even.
	$n=[\ a_kb_k+x_1+1]$;	$1^{(n-1)-\ u_kb_k}$,	$(2x_0-1)^{\ u_kb_k}$,	$(2x_1)^{u_{k-1}b_k+1}$,	k odd.
(2)	$n=[x_1a_k^2+1]$;	$1^{(n-1)-x_1u_ka_k}$,	$(2x_0-1)^{x_1u_ka_k}$,	$(2x_1)^{u_{k-1}a_k}$,	k even.
	$n=[\ a_k^2+1]$;	$1^{(n-1)-\ u_ka_k}$,	$(2x_0-1)^{\ u_ka_k}$,	$(2x_1)^{u_{k-1}a_k}$,	k odd.
(2′)	$n=[x_1b_k^2+1]$;	$1^{(n-1)-x_1u_kb_k}$,	$(2x_0-1)^{x_1u_kb_k}$,	$(2x_1)^{u_{k+1}b_k}$,	k even.
	$n=[\ b_k^2+1]$;	$1^{(n-1)-\ u_kb_k}$,	$(2x_0-1)^{\ u_kb_k}$,	$(2x_1)^{u_{k+1}b_k}$,	k odd.
(3)	$n=[x_1c_k^2+1]$;	$1^{(n-1)-x_1u_kc_k}$,	$(2x_0-1-2x_1)^{x_1u_kc_k}$,	$(2x_1)^{u_{k+1}c_k}$,	k even.
	$n=[\ c_k^2+1]$;	$1^{(n-1)-\ u_kc_k}$,	$(2x_0-1-2x_1)^{\ u_kc_k}$,	$(2x_1)^{u_{k+1}c_k}$,	k odd.
(3′)	$n=[x_1d_k^2+1]$;	$1^{(n-1)-x_1u_kd_k}$,	$(2x_0-1-2x_1)^{x_1u_kd_k}$,	$(2x_1)^{u_{k-1}d_k}$,	k even
	$n=[\ d_k^2+1]$;	$1^{(n-1)-\ u_kd_k}$,	$(2x_0-1-2x_1)^{\ u_kd_k}$,	$(2x_1)^{u_{k-1}d_k}$,	k odd.
(4)	$n=[x_1d_kc_k+x_1+1]$;	$1^{(n-1)-x_1u_kc_k}$,	$(2x_0-1-2x_1)^{x_1u_kc_k}$,	$(2x_1)^{u_{k-1}c_k+1}$,	k even.
	$n=[\ d_kc_k+x_1+1]$;	$1^{(n-1)-\ u_kc_k}$,	$(2x_0-1-2x_1)^{\ u_kc_k}$,	$(2x_1)^{u_{k-1}c_k+1}$,	k odd.
(4c)	$n=[x_1c_kd_k+x_1+1]$;	$1^{(n-1)-x_1u_kd_k}$,	$(2x_0-1-2x_1)^{x_1u_kd_k}$,	$(2x_1)^{u_{k+1}d_k+1}$,	k even.
	$n=[\ c_kd_k+x_1+1]$;	$1^{(n-1)-\ u_kd_k}$,	$(2x_0-1-2x_1)^{\ u_kd_k}$,	$(2x_1)^{u_{k+1}d_k+1}$,	k odd.
(5)	$n=[e_{k+1}e_{k-1}+x_0(x_1-x_0+1)+1]$;	$1^{(n-1)-u_{k+1}e_{k-1}-x_1+x_0}$,	$(2x_0-1)^{u_{k+2}e_{k-1}+x_1-x_0+1}$,	$(2x_1-2x_0+1)^{u_ke_{k-1}+1}$,	k even.
	$n=[e_k^2+1]$;	$1^{(n-1)-u_ke_k}$,	$(2x_0-1)^{u_{k+1}e_k}$,	$(2x_1-2x_0+1)^{u_{k-1}e_k}$,	k odd.
(5′)	$n=[f_{k-1}f_{k+1}+x_0(x_1-x_0+1)+1]$;	$1^{(n-1)-u_{k-1}f_{k+1}-x_1+x_0}$,	$(2x_0-1)^{u_{k-2}f_{k+1}+x_1-x_0+1}$,	$(2x_1-2x_0+1)^{u_kf_{k+1}+1}$,	k even.
	$n=[f_k^2+1]$;	$1^{(n-1)-u_kf_k}$,	$(2x_0-1)^{u_{k-1}f_k}$,	$(2x_1-2x_0+1)^{u_{k+1}f_k}$,	k odd.
(6)	$n=[f_{k+1}e_{k-1}+(x_1-x_0)^2+1]$;	$1^{(n-1)-u_{k+1}e_{k-1}-x_1+x_0}$,	$(2x_0-1)^{u_ke_{k-1}}$,	$(2x_1-2x_0+1)^{u_{k+2}e_{k-1}+x_1-x_0}$,	k even.
	$n=[f_ke_k+x_1+1]$;	$1^{(n-1)-u_ke_k}$,	$(2x_0-1)^{u_{k-1}e_k+1}$,	$(2x_1-2x_0+1)^{u_{k+1}e_k+1}$,	k odd.
(6c)	$n=[e_{k-1}f_{k+1}+(x_1-x_0)^2+1]$;	$1^{(n-1)-u_{k-1}f_{k+1}-x_1+x_0}$,	$(2x_0-1)^{u_kf_{k+1}}$,	$(2x_1-2x_0+1)^{u_{k-2}f_{k+1}+x_1-x_0}$,	k even.
	$n=[e_kf_k+x_1+1]$;	$1^{(n-1)-u_kf_k}$,	$(2x_0-1)^{u_{k+1}f_{k+1}}$,	$(2x_1-2x_0+1)^{u_{k-1}f_k+1}$,	k odd.

$a_i=x_0u_i+u_{i-2}$, all i. $\quad c_i=(x_0-2)u_i-u_{i-2}$, all i. $\quad e_i=x_0(u_{i+1}-u_{i-1})+(u_{i-1}+u_{i-2})$, i odd.

$b_i=x_0u_i+u_{i+2}$, all i. $\quad d_i=(x_0-2)u_i-u_{i+2}$, all i. $\quad f_i=x_0(u_{i-1}-u_{i+1})+(u_{i+1}+u_{i+2})$, i odd.

17. A necessary and sufficient condition that a characteristic be geometric is that there exist a set of T_2's with which it may be successively compounded so that the resultant is the linear characteristic. If the characteristic is geometric its conjugate may be found by compounding the linear characteristic (written so as to contain a basis point of multiplicity zero to correspond to each basis point of the given characteristic) with the same set of T_2's taken in the same order. Triadics $1c$, $4c$, and $6c$ may be derived in this way as the conjugates of triadics 1, 4, and 6, while all the others in the preceding table are self-conjugate.

Unfortunately the algebra involved in the investigations of these triadics is long and complicated, so that we shall merely sketch the outline of the procedure, using triadic 1 to illustrate it. Let the linear characteristic be written under, and in the same form as, triadic 1. Let P be the single point of the first group (see paragraph 2). Compound with T_2's having two basis points in the second group so that this group is reduced to a group of simple points. (One T_2 will have only one basis point in the second group and one general.) Reduce the multiplicity of the third group by compounding with T_2's having two basis points in it. If the multiplicity of the new third group is repeatedly reduced in the same way it is found that the formulas for all of the multiplicities change according to a certain simple law, so that the formulas for the multiplicities resulting from an arbitrary number of such reductions can be calculated directly. The number of such reductions needed is approximately k. When the triadic has been reduced in this way to the simplest possible form it is finally compounded with a set of T_2's which obliterates the group of simple points that has been carried along, and the result is the linear characteristic. In the meantime the same sets of T_2's have operated on the original linear characteristic to produce triadic $1c$. Algebraic simplifications in this procedure are made with the aid of relations (B) and (C).

18. The conjugates of triadics 1, 4, and 6 may be formally obtained by substituting $-k$ for k in these triadics when they are written explicitly in terms of the u's. Triadics $2'$, $3'$, and $5'$ were obtained by this same formal operation when applied to triadics 2, 3, and 5 respectively.

19. In paragraph 14 the case $k=1$ was not considered. Triadics may be obtained from (5), paragraph 3, either by making one of the base numbers vanish or by making two of the multiplicities equal. Triadics 7, 8, 9, 10, and 11 result respectively from the conditions $2x_{01}-1-x_{12}=0$, $x_{12}=0$, $1+2x_{11}-x_{12}=0$, $x_{01}-1=1$, and $1+x_{01}x_{11}-x_{12}=x_{01}$. Their conjugates are easily derived by Montesano's method. We write merely x_0 for x_{01}, x_1 for x_{11}, and x_2 for x_{12}.

(7)	$n = [x_1x_0 - x_0 + 2]$;	$1^{x_1x_0-2x_0+2}$,		$(2x_1 - 2x_0 + 2)^{x_0}$,		$(2x_0 - 1)^{x_0-1}$.
(7c)	$n = [x_1x_0 - x_0 + 2]$;	$1^{x_1x_0-x_1-x_0+1}$,		$(2x_1 - 2x_0 + 2)^{1}$,		$(2x_0 - 1)^{x_1}$.
(8)	$n = [x_1x_0 + x_0 + 1]$;	$1^{x_1x_0+1}$,		$(2x_1 + 1)^{x_0}$,		$(2x_0 - 1)^{1}$.
(8c)	$n = [x_1x_0 + x_0 + 1]$;	$1^{x_1x_0-x_1+x_0}$,		$(2x_1 + 1)^{1}$,		$(2x_0 - 1)^{x_1+1}$.
(9)	$n = [x_1x_0 - 2x_1 + x_0]$;	$1^{x_1x_0-2x_1}$,		$(2x_1 + 1)^{x_0-1}$,		$(2x_0 - 2x_1 - 2)^{1}$.
(9c)	$n = [x_1x_0 - 2x_1 + x_0]$;	$1^{x_1x_0-3x_1+x_0-1}$,		$(2x_1 + 1)^{x_1}$,		$(2x_0 - 2x_1 - 2)^{x_1+1}$.
(10)	$n = [2x_1 - x_2 + 3]$;	$1^{2x_1-x_2+1}$,		$(1 + 2x_1 - x_2)^{2}$,	${x_2}^{1}$,	$(3 - x_2)^{1}$.
(10c)	$n = [2x_1 - x_2 + 3]$;	$1^{x_1-x_2+2}$,		$(1 + 2x_1 - x_2)^{1}$,	${x_2}^{x_1}$,	$(3 - x_2)^{x_1+1}$.
(11)	$n = [2x_0]$;	1^{x_0},	$(2x_1 + x_0 - x_1x_0)^{x_0}$,	$(1 + x_1x_0 - x_0)^{x_0-1}$,		$(3x_0 - x_1x_0 - 2)^{1}$.
(11c)	$n = [2x_0]$;	$1^{2x_0-x_1-1}$,	$(2x_1 + x_0 - x_1x_0)^{1}$,	$(1 + x_1x_0 - x_0)^{x_1}$,		$(3x_0 - x_1x_0 - 2)^{x_1+1}$.

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