H. Umemura
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ON THE MAXIMAL CONNECTED ALGEBRAIC SUBGROUPS OF THE CREMONA GROUP I

HIROSHI UMEMURA

This paper is a continuation of the two preceding papers [12], [13] where the classification of the de Jonquières type subgroups in the Cremona group of 3 variables is promised. However the classification of such subgroups is postponed until the article in preparation "On the maximal connected algebraic subgroups of the Cremona group II". The purpose of this paper is to establish a general method to study algebraic subgroups in the Cremona group of n variables and to illustrate how it works and leads to the classification of Enriques (Theorem (2.25)) when applied to the 2 variable case. This method gives us also the classification of the maximal connected algebraic subgroups of the Cremona group of 3 variables. The reason why we dare to write a new proof of the notorious Enriques Theorem is as follows. The case of 3 variables is rather complicated and we are sometimes obliged to indicate only the results and the way how to prove them without going into the detailed calculations if they are done quite similarly as in the 2 variable case. Hence we consider the best way to understand our classification in the 3 variable case is to read a complete proof in the 2 variable case beforehand. This is a raison d'être of a new proof of the Enriques Theorem and hence of this paper. Our method will be applied for the 4 variable case too.

As in the preceding papers, we work over an algebraically closed field of characteristic 0. All algebraic groups are connected and when we speak of a law chunk of algebraic operation (G, X), X is irreducible. 1 denotes either the unit element of a group or the group consisting only of the unit element.

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§1. Some results in the study of algebraic subgroups of the Cremona group

Let X be an algebraic variety defined over an algebraically closed field k of characteristic 0. We study the group of the birational automorphisms of X. We are particularly interested in connected algebraic subgroups of the group of the birational automorphisms of X. When we speak of the group of the birational automorphisms of X and of a morphism of an algebraic group onto that group, we do not consider the group of the birational automorphisms merely as an abstract group. There is an algebraic structure which makes the group of the birational automorphisms something like an algebraic group. This structure is most naturally expressed if we make a functorial interpretation.

We denote by Autbirat X a group functor on the category of k-schemes; for a k-scheme Z, Autbirat $X(Z) = Z$-pseudo automorphisms of $Z \times X$ (cf. Demazure [3]). Roughly speaking, Autbirat $X(Z) = Z$-birational automorphisms of $Z \times X$. The group of k-valued points Autbirat $X(k)$ is nothing but the 'group of the k-birational automorphisms of X. In general, this functor is far from being representable.

The following theorem clarifies the notion of an algebraic morphism of an algebraic group to Autbirat X.

THEOREM (1.1). *Let G be an algebraic group and X an algebraic variety. Then there is a* 1–1 *correspondence between the following*:

(1) *Morphisms of group functors* $G \to$ Autbirat X.

(2) *Algebraic pseudo-operations* (G, X).

(3) *Algebraic operations* (G, X') + *birational isomorphism* $f\colon X' \dashrightarrow X$ *modulo equivalence relation* $\sim$ (*).

The correspondence (1) $\leftrightarrow$ (2) is Proposition 4, p. 515, Demazure [4]. (2) $\to$ (3) is a well known theorem of Weil (Theorem, p. 375, Weil [15]). (3) $\to$ (2) is evident (cf. Umemura [12]).

(*) $(G, X'_1) + f_1$ is equivalent to $(G, X'_2) + f'_2$ if and only if the diagram

$$\begin{tikzcd} G \times X'_1 \arrow[r] \arrow[d, dashed, "\mathrm{Id} \times f_1"] & X'_1 \arrow[d, dashed, "f_1"] \\ G \times X & X \\ G \times X'_2 \arrow[r] \arrow[u, dashed, "\mathrm{Id} \times f_2"] & X_2 \arrow[u, dashed, "f_2"] \end{tikzcd}$$

is commutative.

Definition (1.2). An algebraic subgroup G of Autbirat X is a subgroup functor of Autbirat X representable by an algebraic variety.

The following result is also found in Demazure [3] (Proposition 3, p. 513).

Proposition (1.3). *Let X be a variety, G an algebraic group and φ a morphism of G to* Autbirat X. *Then the* $\operatorname{Ker}\varphi$ *is represented by a closed subgroup of G.*

We need not only algebraic subgroups but also morphisms $G \to$ Autbirat X with finite kernels.

Definition (1.4). Let (G, X) be an algebraic pseudo-operation. We say (G, X) is effective (resp. almost effective) if the kernel of the induced morphism $G \to$ Autbirat X is 1 (resp. a finite group scheme). It follows from Theorem (1.1).

Proposition (1.5). *There is a* 1:1 *correspondence between the following two sets*:

{*connected algebraic subgroups in* Autbirat X} / *conjugacy* $\xleftrightarrow{1:1}$ {*effective algebraic operations* $(G, X') \mid G$ *is a connected algebraic group, X' is birationally isomorphic to X*} / *isomorphism of law chunks of algebraic operation.*

See Umemura [13].

Let G be a connected algebraic subgroup of Autbirat X. Let C be the set of all the algebraic subgroups of Autbirat X conjugate to G. C will be simply called a conjugacy class of connected algebraic subgroups in Autbirat X. There corresponds to C, by Proposition (1.5), an equivalence class of algebraic operations. Any algebraic operation (G, X) of this equivalence class is called an effective realization (or simply a realization) of the conjugacy class C. An operation (G', X) such that there exists a morphism of algebraic operation $(\varphi, \mathrm{Id})\colon (G', X) \to (G, X)$ with $\operatorname{Ker}\varphi$ finite is said to be an almost effective realization of the conjugacy class C.

Lemma (1.6). *Let X be an algebraic variety and G an (connected) algebraic group contained in* Autbirat X. *The following are equivalent*:

(1) *There exists a realization (G, X') of the conjugacy class of G in* Autbirat X *such that G has a non-empty open orbit on X'.*

(2) *For any realization (G, X') of the conjugacy class of G in* Autbirat X, *G has a non-empty open orbit on X'.*

Proof. Since (2) $\Rightarrow$ (1) is trivial, we prove (1) $\Rightarrow$ (2). Let (G, X'), (G, X'') be realizations of G and we assume (G, X') has a non-empty open orbit. It follows from the definition there exists an isomorphism $(\varphi, f)\colon (G, X') \to (G, X'')$ of law chunks of algebraic operations. Hence by Corollary p. 404 Rosenlicht [7], (G, X'') also has a non-empty open orbit.

DEFINITION (1.7). Let X, G be as in Lemma (1.6). If G satisfies one of the condition of Lemma (1.6), G is said to be generically transitive. Otherwise, G is said to be generically intransitive.

The effectiveness of an algebraic operation (G, X) sometimes restricts the group G.

LEMMA (1.8). *Let (G, X) be an effective algebraic operation. If (G, X) is generically transitive and G is abelian, then* $\dim G = \dim X$.

Proof. Let G/H be the open orbit in X. The morphism $G \to \operatorname{Aut} X$ factors through $G \to \operatorname{Aut} G/H$. Since G/H is open in X, the homogeneous space $(G, G/H)$ is effective. As G is abelian, this shows $H = 1$ and $\dim G = \dim G/H = \dim X$.

DEFINITION (1.9). Let X be a variety and C_1, C_2 be conjugacy classes of connected algebraic subgroups of $\operatorname{Autbirat} X$. We say C_1 is a subgroup of C_2 if there exist algebraic groups G_i $(i = 1, 2)$ such that $G_1 \subset G_2 \subset \operatorname{Autbirat} X$ and such that G_i belongs to the conjugacy class C_i.

PROPOSITION (1.10). *Let X be a variety and C_i $(i = 1, 2)$ conjugacy classes of connected algebraic subgroups of* $\operatorname{Autbirat} X$. *Let (G_i, X_i) be realizations of C_i for $i = 1, 2$. Then, the following are equivalent.*

(1) *C_1 is a subgroup of C_2.*

(2) *There exists a morphism $(\varphi, f)\colon (G_1, X_1) \to (G_2, X_2)$ of law chunks of algebraic operations such that f is birational.*

Proof. (1) $\Rightarrow$ (2). In fact, there exist birational maps $h_i\colon X_i \to X$, $i = 1, 2$ and the condition (1) implies there exists a birational automorphism $h\colon X \to X$ such that $h(h_1 G_1 h_1^{-1})h^{-1} \subset h_2 G_2 h_2^{-1}$. This means there exists a (rational hence regular) morphism $\varphi\colon G_1 \to G_2$ such that $(\varphi_1, h_2^{-1} \circ h \circ h_1)\colon (G_1, X_1) \to (G_2, X_2)$ is a morphism of algebraic law chunks with birational $h_2^{-1} \circ h \circ h_1$.

(2) $\Rightarrow$ (1). Let us fix a birational map $h_2\colon X_2 \to X$. Then, $h_2 f G_1 f^{-1} h_2^{-1}$ is an algebraic subgroup of C_1 and $h_2 G_2 h_2^{-1}$ is an algebraic subgroup of C_2. The inclusion $f G_1 f^{-1} \subset G_2$ implies the inclusion $h_2 f G_1 f^{-1} h_2^{-1} \subset h_2 G_2 h_2^{-1}$.

DEFINITION (1.11). The Cremona group Cr_n of n variable is, by definition, Autbirat $\boldsymbol{P}^n$.

It follows from the definition that $\mathrm{Cr}_n(k)$ is the group of the k-automorphisms of the rational function field of n-variables. As we proved in [13], algebraic subgroups of the Cremona groups are linear. Hence, from now on we assume all the algebraic groups are linear.

In the proof of Theorem (3.7) Umemura [13], it is important to observe the orbits of a normal subgroup, e.g., the unipotent radical, the center of the unipotent radical. The idea of studying the orbits of a normal subgroup looks naive but it is rather powerful. This method is so useful that we use it repeatedly. Here is the first example of the application.

LEMMA (1.12). *Let $(G, G/H)$ be an algebraic homogeneous space and n the dimension of G/H. We assume the operation of G is effective. If there is a normal subgroup N of G isomorphic to $\boldsymbol{G}_a^{\oplus m}$ such that N has an n dimensional orbit, then* (1) $m = n$, (2) *G/H is isomorphic to the affine space $\boldsymbol{A}^n$, and* (3) *$(G,G/H)$ is a suboperation of the n dimensional affine transformation $(GTA_n, \boldsymbol{A}^n)$,* (4) *$G$ is the semi-direct product $N\cdot H$.*

Proof. Let X be the n dimensional N-orbit. Then X is an open subset. Since N is normal, for any $g \in G$, gX is also an n dimensional N-orbit. Hence gX coincides with X, i.e. X is G-invariant and $X = G/H$. Since N operated on $X = G/H$ transitively and effectively and since N is abelian, m should be equal to n by Lemma (1.8) and X is isomorphic to $\boldsymbol{G}_a^{\oplus n}$, hence to the affine space $\boldsymbol{A}^n$ and N operates on $X = G/H \simeq \boldsymbol{A}^n$ as translation group. As N operates transitively on G/H, we get $G = NH$. Since the operation of N is effective, $N \cap H = 1$ hence G is a semi-direct product. Now, it remains to show that H is a subgroup of the affine translation. Let $g \in H$, $n \in N$, then the result of the operation of g on the coset nH is $gnH = gn(g^{-1}g)H = gng^{-1}H$. Hence the operation of H on $\boldsymbol{A}^n$ is linear.

COROLLARY (1.13). *Let $(G, G/H)$ be an effective homogeneous space as in Lemma* (1.12). *If the center U_Z of the unipotent radical of G has n dimensional orbit* (*here* $n = \dim G/H$), *then* (1) $\dim U_Z = n$, (2) *G/H is isomorphic to the affine space $\boldsymbol{A}^n$,* (3) *$(G, G/H)$ is a suboperation of the n dimensional affine transformation $(GTA_n, \boldsymbol{A}^n)$,* (4) *the unipotent radical U is abelian, namely $U = U_Z$,* (5) *H is reductive and* (6) *G is a semi-direct product $G = H\cdot U_Z \simeq H\cdot \boldsymbol{G}_a^{\oplus n}$.*

Proof. The assertions (1), (2) and (3) follow from Lemma (1.12). To prove (4), we need

LEMMA (1.14). *Let U be a (connected) unipotent algebraic group and U' a closed normal subgroup. If the quotient group U/U' is* 1 *dimensional (hence isomorphic to G_a), then the exact sequence*

$$1 \longrightarrow U' \longrightarrow U \xrightarrow{p} U/U' \longrightarrow 1$$

splits. Namely there exists a closed subgroup K of U such that the restriction $p|_K : K \to U/U'$ is an isomorphism.

Before we start the proof, we notice the assertion of the Lemma is evidently false if $\operatorname{char} k > 0$. If U is abelian, the result is well-known (Corollaire, p. 172, Serre [8]). We prove the lemma by induction on the dimension of U. If $\dim U = 1$, a non-zero homomorphism of G_a to G_a is an isomorphism because $\operatorname{char} k = 0$. Let us assume the Lemma holds for $\dim U \leq m$. Let U, U' be as in Lemma (1.14) with dimension $U = m + 1$. The unipotent group has the non-trivial center U_Z with $\dim U_Z \geq 1$. If $p|_{U_Z}$ is surjective, as we have seen above the exact sequence

$$1 \longrightarrow \operatorname{Ker} p|_{U_Z} \longrightarrow U_Z \longrightarrow U/U' \longrightarrow 1$$

splits and we have nothing to prove. Hence we may assume $p(U_Z) = 1$. This is equivalent to say that the diagram

$$\begin{tikzcd}
 & & 1 \arrow[d] & & \\
 & & U_Z \arrow[d] & & \\
1 \arrow[r] & U' \arrow[r] & U \arrow[r, "p"] \arrow[d, "q"'] & U/U' \arrow[r] \arrow[dl, dashed, bend left, "s"] & 1, \\
 & & U/U_Z \arrow[ur, "r"] \arrow[d] & & \\
 & & 1 & &
\end{tikzcd}$$

commutes where q, r are canonical maps. It follows from the induction hypothesis r has a section s. It is sufficient to show q has a section over $s(U/U')$, i.e. the exact sequence $1 \to U_Z \to q^{-1}(s(U/U')) \to s(U/U') \to 1$ splits.

Since U_Z is abelian, we have to show $H^2(G_a, U_Z) = 0$ for the trivial G_a-module U_Z. Let more generally W be a finite dimensional vector group on which G_a operates trivially, then $H^2(G_a, W) = 0$ since W is the direct sum of G_a's and $H^2(G_a, G_a) = 0$ (See Proposition 8, p. 172, Serre [8]).

Let us come back to the proof of the assertion (4) in Corollary (1.13). Consider the exact sequence,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U_Z & \longrightarrow & U & \longrightarrow & U/U_Z & \longrightarrow & 1\,. \\ & & & & \cup & & \cup & & \\ & & & & K & \xrightarrow{\sim} & G_a & & \end{array}$$

Assume $U/U_Z \neq 1$. Since U/U_Z is unipotent, it contains G_a. The exact sequence restricted on this G_a splits by Lemma (1.14). Hence, there exists a subgroup K of U lying over the G_a. Then the subgroup $U_Z \cdot K$ is unipotent and abelian. Hence $U_Z \cdot K$ is isomorphic to $G_a^{\oplus n+1}$. On the other hand, in the proof of the Lemma, we proved G/H is homogeneous space of U_Z hence in particular, a homogeneous space of $U_Z \cdot K$. Since $U_Z \cdot K$ is abelian and of dimension $n+1$, the operation of $U_Z \cdot K$ on its homogeneous space of dimension n can not be effective by Lemma (1.8). This contradicts the assumption that the operation of G is effective. Hence U coincides with U_Z. Now the assertions (5), (6) follow from Lemma (1.12).

For the definitions of primitive, imprimitive and de Jonquières type transformations, we refer the reader to Umemura [12], [13]. But we recall the following properties because they are basic and used frequently in the sequel.

(1.15) Let G be an analytic group and H a closed analytic subgroup. Then an analytic operation $(G, G/H)$ is primitive if and only if the Lie algebra of H is a maximal Lie subalgebra of the Lie algebra of G (Proposition (1.7), Umemura [13]).

(1.16) Let G be an (connected) algebraic group and H be a closed subgroup. Then an algebraic operation $(G, G/H)$ is of de Jonquières type if and only if there exists a closed algebraic subgroup K of G such that $H \subset K \subsetneqq G$ and $\dim H < \dim K$.

The following theorem is an algebraic version of a theorem of Morozoff [6].

THEOREM (1.17). *Let $(G, G/H)$ be an effective algebraic operation with $\dim G/H \geqslant 2$*[*]. *We assume that $(G, G/H)$ is primitive, i.e., the associated*

[*] If $\dim G/H = 1$, $(G, G/H)$ is a suboperation of (PGL_2, P^1).

analytic operation $(G, G/H)^{an}$ *is primitive.*

(1) *If* G *is not semi-simple, then* (i) G/H *is isomorphic to an affine space,* (ii) $(G, G/H)$ *is (algebraically) an suboperation of the affine transformation group,* (iii) *the unipotent radical* U *of* G *is abelian and* H *is a reductive part of* G *and* (iv) U *is an irreducible* H*-module.*

(2) *If* G *is semi-simple, then the Lie algebra* $\mathfrak{g}$ *of* G *is either simple or isomorphic to the direct product* $\mathfrak{g}_1 \times \mathfrak{g}_1$ *of the two copies of a simple Lie algebra* $\mathfrak{g}_1$.

Proof. If G is not semi-simple, we notice first that G cannot be reductive. For, otherwise G has a nontrivial center Z and we can find in Z a 1 dimensional normal subgroup N of G. Then $K = NH$ satisfies the condition of (1.16) and $(G, G/H)$ is of de Jonquières type hence $(G, G/H)^{an}$ is imprimitive. Let U be the unipotent radical and U_Z be its center. U_Z is a normal subgroup of positive dimension. Therefore, if $U_Z \cdot H \subsetneq G$, $K = U_Z \cdot H$ satisfies the condition of (1.16) and $(G, G/H)$ is of de Jonquières type and $(G, G/H)^{an}$ is imprimitive. Hence we may assume $U_Z H = G$, namely U_Z has an open orbit. If U were not irreducible, then there would exist a proper H-invariant space U' and $K = U' \cdot H$ would satisfy the condition (1.16) and $(G, G/H)$ would be of de Jonquières type hence imprimitive. The assertions (1) (i), (ii), (iii) follow from Corollary (1.13).

Now, we assume G is semi-simple. Let $\tilde{G}$ be the universal covering of G. Then, the Lie algebra of $\tilde{G}$ is isomorphic to G and $\tilde{G}$ is simply connected. Let $\varphi: \tilde{G} \to G$ be the covering map. It is known that φ is finite and algebraic. Let $\tilde{H} = \varphi^{-1}(H)$. Then the algebraic operation $(\tilde{G}, \tilde{G}/\tilde{H})$ is primitive. Though the algebraic operation $(\tilde{G}, \tilde{G}/\tilde{H})$ is not effective but almost effective. Let $\tilde{G} = G_1^{s_1} \times G_2^{s_2} \times \cdots \times G_r^{s_r}$ where G_i is simple, $s_i \geq 1$ is an integer for $1 \leq i \leq r$ and we assume that if $1 \leq i < j \leq r$, G_i is not isomorphic to G_j. First, let us show $r = 1$. Assume $r \geq 2$. Since the Lie algebra $\mathfrak{h}$ of H is maximal in $\mathfrak{g}$, by (1.15) the projection $p_j: H \to G_j^{s_j}$ is either surjective or $p_{j*}(\mathfrak{h})$ is a maximal Lie subalgebra in $\mathfrak{g}_j^{s_j}$ for $1 \leq j \leq r$ where $\mathfrak{g}_j$ is the Lie algebra of G_i. This latter case never happens as we assume $r \geqslant 2$.

In fact if there exist a j such that $p_{j*}(\mathfrak{h})$ is maximal in $\mathfrak{g}_i^{s_i}$, then we have $\mathfrak{h} \subset p_{j*}^{-1} p_{j*}(\mathfrak{h}) \subsetneq \mathfrak{g}$. As $r \geq 2$, there exists an $i \neq j$. Since the operation $(\tilde{G}, \tilde{G}/\tilde{H})$, is almost effective, h does not contain non-zero ideal. In particular, the ideal $0 \times \cdots \times 0 \times \mathfrak{g}_i^{s_i} \times 0 \times \cdots \times 0$ is not contained in

$\mathfrak{h}$ but is contained in $p_{j*}^{-1}p_{j*}(\mathfrak{h})$. Hence $\mathfrak{h} \subsetneqq p_{j*}^{-1}p_{j*}(\mathfrak{h}) \subsetneqq \mathfrak{g}$ which contradicts the maximality of the Lie subalgebra $\mathfrak{h}$. We have therefore surjective morphisms: $\mathfrak{h} \to \mathfrak{g}_i^{s_i}$ for any i. By Levi's theorem $\mathfrak{h}$ is isomorphic to the semi-direct product $\mathfrak{h} = r + \mathfrak{h}_s$ where r is the radical of $\mathfrak{h}$ and $\mathfrak{h}_s$ is semi-simple. We are given surjective morphisms: $\mathfrak{h}_s \to \mathfrak{g}_i^{s_i}$ for any i. But it is an easy excercise of Lie algebra to conclude $\dim h_s \geq \sum_{i=1}^{r} s_i \dim \mathfrak{g}_i = \dim \mathfrak{g}$ hence $\dim h \geq \dim \mathfrak{g}$. We have proved r should be 1. Now, we may assume $\tilde{G} = G'^s$ where G' is simple and $s \geq 1$ is an integer. We have to show $s \leq 2$. Assume $s \geq 3$. And let $\mathfrak{g}'$ be the Lie algebra of G' and $p = p_{2,3,\cdots,s}$ be the projection of $\tilde{G} = G'^s$ onto the last $s-1$ factors. The same argument as above shows the map $p_*: \mathfrak{h} \to \mathfrak{g}'^{(s-1)}$ is surjective. Since the Lie algebra $\mathfrak{g}'$ of G' is semi-simple, the map $p_*: \mathfrak{h} \to (g')^{s-1}$ splits and there exist a section $i: (\mathfrak{g}')^{s-1} \to \mathfrak{h}$. The composite morphism $p_1 \circ i: (\mathfrak{g}')^{s-1} \to \mathfrak{g}'$ is either 0 or surjective. If $p_1 \circ i$ is zero, $\mathfrak{h}$ contains an ideal $0 \times \mathfrak{g}' \times \cdots \times \mathfrak{g}'$ hence H contains a normal subgroup $1 \times G' \times \cdots \times G'$ of dimension $(s-1) \dim G'$ which contradicts the effectiveness. Thus, $p_1 \circ i: (\mathfrak{g}')^{s-1} \to \mathfrak{g}'$ is surjective. But any non-zero morphism $(\mathfrak{g}')^{s-1} \to \mathfrak{g}'$ is up to automorphism of $\mathfrak{g}'$, the projection onto the j-th factor for a suitable $1 \leq j \leq s-1$. In particular, $p_1 \circ i$ is the projection onto the j-th factor for a suitable $1 \leq j \leq s-1$, up to automorphism of $\mathfrak{g}'$. It follows, then up to automorphism of G'^s, the composite morphism $(G')^{s-1} \xhookrightarrow{i} H \hookrightarrow G'^s$ is nothing but $(p_j, p_1, p_2, \cdots, p_{s-1})$. In particular, H contains a normal subgroup

$$1 \times G' \times \cdots \times G' \times \overset{j+1}{1} \times \cdots \times G'$$

of dimension $(s-2) \dim G' > 0$ which is a contradiction. Hence $s \leq 2$.

We can answer in positive way to the question proposed in [13].

COROLLARY (1.18). *Let G_i $(i = 1, 2)$ be connected algebraic groups and H_i be closed algebraic subgroups of G_i such that the algebraic operation $(G_i, G_i/H_i)$ are effective and primitive. Let $(\varphi, f): (G_1, G_1/H_1)^{an} \to (G_2, G_2/H_2)^{an}$ be an isomorphism of analytic operations. Then the morphism (φ, f) is algebraic, i.e., there exists an isomorphism $(\varphi', f'): (G_1, G_1/H_1) \to (G_2, G_2/H_2)$ of algebraic operations such that $(\varphi', f')^{an} = (\varphi, f)$.*

Proof. Since the algebricity of f follows from that of φ, we have to show φ is algebraic. If G_1 is semi-simple, then the assertion is trivial. For, an analytic homomorphism of a semi-simple algebraic group to an algebraic group is algebraic (Théorème 10, VIII-8, Serre [10]). If G_1 is not semi-simple, then by the proof of Theorem (1.17), G_i is the semi-direct product H_iU_i where H_i is reductive, U_i is the unipotent radical of G_i and U_i is abelian ($i = 1, 2$). Let R_i denote the radical of G_i ($i = 1, 2$). By replacing the reference point of G_2/H_2, we may assume $\varphi(H_1) = H_2$. Since H_1 is reductive, $\varphi|_{H_1}: H_1^{an} \to H_2^{an}$ is algebraic by Théorème 10, VIII-8, Serre [10]. φ induces an isomorphism between the radicals R_1 and R_2 and we have to show $\varphi|_{R_1}$ is algebraic. If $R_1 = U_1$, then $R_2 = U_2$ and the isomorphism $\varphi|_{U_1}: U_1^{an} \to U_2^{an}$ is linear hence algebraic. If $R_1 \supsetneqq U_1$, then $R_1 = U_1\cdot \boldsymbol{G}_m$ and $R_2 = U_2\cdot \boldsymbol{G}_m$. For since, H_1-module U_1 is irreducible, the center of H_1 is at most $\boldsymbol{G}_m$. Since the operation is effective, we may assume the group structure of R_1 is defined by $t\cdot u\cdot t^{-1} = t \times u$ (multiplication of the scalor t on the vector u is denoted by $t \times u$) for $t \in \boldsymbol{G}_m$, $u \in U_1$. We shall describe the restriction $\varphi|_{R_1} = \psi$. Since $\varphi(H_1) = H_2$, $\varphi(\boldsymbol{G}_m) = \boldsymbol{G}_m$. Thus $\psi: R_1 = U_1\cdot \boldsymbol{G}_m \to R_2 = U_2\cdot \boldsymbol{G}_m$ is written as $\psi(u\cdot t) = \psi_1(u)\psi_2(u)\psi_2(t)$ for $u \in U_1$, $t \in \boldsymbol{G}_m$ where $\psi_1: U_1 \to U_2$ is an analytic map and $\psi_2: R_1^{an} \to \boldsymbol{G}_m^{an}$ is an analytic morphism of Lie groups. We shall show $\psi_2(u) = 1$ for any $u \in U_1$. In fact, let $u_1\cdot t_1$, $u_2\cdot t_2 \in R_1$ and write the condition for $\varphi((u_1\cdot t_1)(u_2\cdot t_2)) = \varphi(u_1\cdot t_1)\varphi(u_2\cdot t_2)$. Comparing the $\boldsymbol{G}_m$ component, we get $\psi_2(u_1\cdot(t_1 \times u_2)) = \psi_2(u_1)\psi_2(u_2)$ hence $\psi_2(t_1 \times u_2) = \psi_2(u_2)$. Therefore, for any $t \in \boldsymbol{G}_m$, $u \in U_1$, $\psi_2(t \times u) = \psi_2(u)$ hence $\psi_2(u) = 1$. It follows ψ maps U_1 onto U_2 hence $\psi|_{U_1}$ is linear and φ is algebraic.

As we proved in Umemura [13], in Cr_3 there are few algebraic operations which are imprimitive and not of de Jonquières type. The following theorem is nothing but an abstraction of Theorem (3.7), Umemura [13].

THEOREM (1.19). *Let (G, X) be an effective algebraic operation. If (G, X) is imprimitive (i.e., the associated analytic operation $(G, X)^{an}$ is imprimitive) and if (G, X) is not of de Jonquières type, then (G, X) is generically transitive and G is semi-simple.*

Proof. Let us first notice that $\dim X \geq 2$ because (G, X) is imprimitive. By Umemura [13], a generically intransitive operation is of de Jonquières type if the dimension of the transformation space ≥ 2. Hence we may assume (G, X) is a homogeneous space $(G, G/H)$. By (1.16), it is sufficient to construct a closed subgroup K such that $H \subset K \subsetneqq G$ and $\dim H < \dim K$ under the hypothesis that G is not semi-simple. If G is solvable, then there exists a 1 dimensional closed normal subgroup N of G. Since $(G, G/H)$ is effective, the normal group N is not contained in H and $K =$

HN satisfies our requirement. If G is reductive, we may assume G is not a torus because $G =$ torus case is trivial. Let Z be the center of G. If G is not semi-simple, $\dim Z \geq 1$. Let N be a 1 dimensional closed subgroup of Z. Then N is normal and the normal subgroup N is not contained in H because $(G, G/H)$ is effective. The closed subgroup $K = HN$ satisfies the desired condition. It remains to check the case where G is neither solvable nor reductive. Let U_Z denote the center of the unipotent radical U of G. Since U_Z is normal, U_Z is not contained in H. Hence, if $U_Z \cdot H \neq G$, then it is sufficient to put $K = U_Z H$. Therefore we may assume $U_Z \cdot H = G$. Namely U_Z orbit $U_Z H$ coincides with G/H. It follows from Corollary (1.13) $U = U_Z$ and G is the semi-direct product $U_Z \cdot H$. Let $\mathfrak{g}, \mathfrak{h}, u_Z$ denote the Lie algebras of G, H, U_Z. Therefore $\mathfrak{g} = u_Z + \mathfrak{h}$. By (1.15), the Lie algebra $\mathfrak{h}$ is not maximal. This implies $\mathfrak{h}$-module u_Z is not irreducible. Consequently, the vector group U_Z is not an irreducible H-module. Let $0 \neq W \subsetneqq U_Z$ be an H-invariant submodule. Then $K = W \cdot H$ satisfies our requirement.

COROLLARY (1.20). *Let (G, X) be an effective algebraic operation. We assume the associated analytic operation $(G, X)^{an}$ is imprimitive. If G is not semi-simple, then (G, X) is of de Jonquières type.*

This is a part of Theorem (1.19).

LEMMA (1.21). *Let G be a (connected) reductive algebraic group, (G, X) an algebraic operation, $f: X \to Y$ a morphism of algebraic varieties such that the diagram*

$$\begin{tikzcd} G \times X \arrow[rr, "\varphi"] \arrow[dr, "f \circ p_2"'] & & X \arrow[dl, "f"] \\ & Y & \end{tikzcd}$$

is commutative, where φ is the operation and p_2 is the projection $G \times X \to X$. If, for any closed point $y \in Y$, the operation of G on the fibre $X_y = f^{-1}(y)$ is not almost effective (resp. effective), then the operation (G, X) is not almost effective (resp. effective).

Proof. As the effective case is treated similarly, we prove the Lemma for the almost effective case. Since a normal subgroup of positive dimension of G contains a torus, a torus is contained in a maximal torus and since maximal tori are conjugate, a maximal torus T of G satisfies the hypothesis. Therefore, we may assume G is a torus. Let F be the inverse image of the diagonal of $X \times X$ by the morphism $(\varphi, p_2)\colon G \times X \to X \times X$. F is a closed subscheme of $G \times X$ and a subgroup scheme over X of $G \times X$. Since a morphism is generically flat, there exists a non-empty open subset U of X such that the projection $p_2\colon F \cap G \times U \to U$ is faithfully flat. By Lemma 5, p. 520, Demazure [3], there exists a closed subscheme H of G such that $F = H \times U$. Let $x \in U$ be a closed point. It follows from the assumption, that there exists a closed subgroup K of positive dimension of G operating trivially on the fibre $X_{f(x)}$. This shows K is a subgroup of H and the operation of G is not almost effective.

§ 2. Classification of the maximal connected algebraic subgroups of the Cremona group of two variables

The classification, up to conjugacy, of the maximal connected algebraic subgroups of the Cremona group of two variables Cr_2 was done by Enriques [4]. We apply the results of § 1 to recover the result of Enriques. The method of Enriques is quite different from ours and depends on the classification of linear systems on two dimensional rational varieties. Let us indicate roughly why the classification of the maximal connected algebraic subgroups of the Cremona group is related to the classification of linear systems on rational varieties. Let (G, X) be a realization of a conjugacy class of a connected algebraic subgroups in the Cremona group of n variables. As the theorem of the equivariant resolution of singularity is proved, then by equivariant completion and equivariant Chow's lemma (Sumihiro [11]), we could assume X non-singular and projective. Then, since X is rational, $h^{01}(X) = h^{10}(X) = 0$. Hence the Picard scheme of X is descrete and the algebraic group G leaves any complete linear system on X invariant. A maximal algebraic group of Cr_n is defined to be the group of the birational transformations leaving a linear system invariant.

The advantage of our method is the comprehensiveness (we hope) and to be extendable to the higher dimensional case. The three dimensional case is treated in Umemura [12], [13] and will be completed by our method in Umemura [14].

(A) Classification of the maximal primitive connected algebraic subgroups of Cr_2.

Let (G, X) be a realization of a conjugacy class of the maximal primitive connected algebraic subgroups of Cr_2. Then, by Proposition (1.10), Theorem (1.17) and Corollaire 1, p. 521, Demazure [14], G is semi-simple of rank ≤ 2. Hence, the Lie algebra $\mathfrak{g}$ of G is isomorphic to sl_2, sl_3, $sl_2 \times sl_2$ or G_2. Let X be isomorphic to G/H, then by Umemura [13], the Lie algebra $\mathfrak{h}$ of H has the following properties: (1) $\mathfrak{h}$ is maximal Lie subalgebra of $\mathfrak{g}$, (2) h contains no non-zero ideals of $\mathfrak{g}$, (3) $\dim \mathfrak{g} - \dim \mathfrak{h} = 2$. It is an easy exercise to see that G_2 contains no Lie subalgebra of codimension 2. The only one possible such G and H is $(PGL_3, \boldsymbol{P}_2)$. Hence, any primitive connected algebraic subgroup of Cr_2 is, up to conjugacy, is contained in $(PGL_3, \boldsymbol{P}_2)$.

(B) Classification of the imprimitive connected algebraic groups which are not of de Jonquières type.

Before we begin the classification, let us notice the following Lemma which is almost evident but will be often used.

LEMMA (2.1). *Let (G, X) be a realization of a conjugacy class C of the connected algebraic groups in Cr_2. Then, the following are equivalent:*

(1) *The conjugacy class C is of de Jonquières type.*

(2) *The algebraic operation (G, X) is of de Jonquières type.*

(3) *There exists a morphism $(\varphi, f)\colon (G, X) \to (PGL_2, \boldsymbol{P}^1)$ of law chunks of algebraic operation such that $\varphi(G) \neq 1$.*

If we assume the realization (G, X) is a homogeneous space $(G, G/H)$, then the following condition is equivalent to the preceding conditions (1), (2), (3):

(4) *There exists a morphism $(\varphi, f)\colon (G, G/H) \to (PGL_2, \boldsymbol{P}^1)$ of algebraic operations with $\varphi(G) \neq 1$.*

Proof. The equivalence of (1) and (2) is the definition. Let (G', X') be an algebraic operation such that X' is a rational curve. Then (G', X') is considered as a suboperation of $(PGL_2, \boldsymbol{P}^1)$ and a unirational curve is rational. This shows the equivalence of (2) and (3). The equivalence of (3) and (4) follows from Corollary, p. 404, Rosenlicht [7].

The following Proposition is similar to Theorem (3.7) Umemura [13] and proved by the same method.

THEOREM (2.2). *Let (G, X) be a realization of a conjugacy class of the connected algebraic subgroups in Cr_2. Let $\mathfrak{g}$ be the Lie algebra of G. If*

(G, X) is imprimitive and if $\mathfrak{g}$ is not isomorphic to sl_2, then (G, X) is of de Jonquières type.

Proof. By Theorem (1.19), we may assume that (G, X) is generically transitive and G is semi-simple. Therefore we assume (G, X) is a homogeneous space $(G, G/H)$. Since any 1 dimensional effective analytic law chunk is contained in (PGL_2, P^1), it follows from the hypothesis that there exists a morphism of law chunks of analytic operation $(\varphi, f): (G, G/H)^{an} \to (PGL_2, P^1)^{an}$ with non-trivial φ. Hence, the Lie algebra $\mathfrak{g}$ contains sl_2 as a direct factor. If $\mathfrak{g}$ is not isomorphic to sl_2, then the $\ker \varphi$ contains an ideal $\mathfrak{k}$ of $\mathfrak{g}$ such that $\mathfrak{k}$ is a simple Lie algebra. Since $\mathfrak{k}$ is simple, there exists a normal closed algebraic subgroup K of G corresponding to $\mathfrak{k}$ by Theorems 15, p. 177, Chevalley [2]. The orbits of K are one dimensional because $k \subset \ker \varphi$. Hence $KH \subsetneq G$. Since the operation $(G, G/H)$ is effective, the normal subgroup K is not contained in H and $\dim H < \dim KH$. Therefore by (1.16), $(G, G/H)$ is of de Jonquières type.

COROLLARY (2.3). *Let (G, X) be a realization of a conjugacy class of the connected algebraic groups in* Cr_2. *If (G, X) is imprimitive and not of de Jonquières type, then (G, X) is isomorphic to $(SO_3, SO_3/D_\infty)$ as an algebraic law chunk of operation (or in usual language, birationally isomorphic to).* The definition of D_∞ is given in the proof.

Proof. By Theorem (1.19) and Theorem (2.2), (G, X) is isomorphic to $(G, G/H)$ as an algebraic law chunk of operation and G is isomorphic to SL_2 or SO_3. There is a morphism $\varphi: SL_2 \to SO_3$ of degree 2. Hence in either case, G/H is isomorphic to SL_2/H' for an appropriate 1 dimensional closed algebraic subgroup H' of SL_2. It is easy to see that a 1 dimensional closed algebraic subgroup H' of SL_2 is conjugate to one of the following:

(1) $$T = \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in SL_2\right\},$$

(2) $$D_\infty = \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix} \in SL_2\right\},$$

(3) $$U_n = \left\{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in SL_2 \,|\, a^n = 1\right\}, \qquad n = 1, 2, \cdots.$$

Hence the operation $(G, G/H)$ is isomorphic to $(SL_2, SL_2/H')$ or $(SO_3, SO_3/\varphi(H'))$ as an algebraic operation. If H' is T or U_n $(n = 1, 2, \cdots)$, then

$$H' \subsetneqq B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in SL_2 \right\}$$

and the operation $(G, G/H)$ is of de Jonquières type by (1.16). Hence $H' = D_\infty$ and D_∞ is not contained in a Borel subgroup. Therefore $(SL_2, SL_2/D_\infty)$ is not of de Jonquières type. But the operation $(SL_2, SL_2/D_\infty)$ is not effective. Hence $(G, G/H)$ is isomorphic to $(SO_3, SO_3/\varphi(D_\infty))$ as an algebraic operation. Since $\varphi(D_\infty)$ is isomorphic to D_∞, $\varphi(D_\infty)$ is, by abuse of notation, denoted by D_∞.

PROPOSITION (2.4). *The conjugacy class in* Cr_2 *of the almost effective realization* $(SL_2, SL_2/D_\infty)$ *is a subgroup of the conjugacy class of the realization* (PGL_3, P_2), *hence is not maximal among the connected algebraic subgroups in* Cr_2.

Proof. Let V be the irreducible representation of degree 2 of SL_2. Let u, v be a base of V such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2 \quad \text{transforms} \quad (u, v) \longmapsto (au + cv,\ bu + dv)\ .$$

Let us consider the irreducible representation $S^2(V)$ of degree 3 and projectify the SL_2 operation on $P(S^2(V))$. Then the stabilizer at $uv \in P(S^2(V))$ coincides with D_∞. Therefore we get a morphism $(SL_2, SL_2/D_\infty) \to (PGL_3, P(S^2(V)))$ and the Proposition follows from Proposition (1.10).

(C) Classification of the maximal connected algebraic subgroups of de Jonquières type in Cr_2.

Case (C-tr). First we treat generically transitive operations. Hence, we assume that all realizations are homogeneous space $(G, G/H)$.

Case (C-tr-1). G is neither reductive nor solvable.

Let U denote the unipotent radical of G and U_Z the center of U. It follows from the hypothesis that $\dim U_Z \geq 1$. If the dimension of the orbit $U_Z H \subset G/H$ is 2, then $(G, G/H)$ is a suboperation of the 2 dimensional affine transformation group by Theorem (1.13) and is not maximal. Hence, we may assume that the dimension of the orbit $U_Z H \subset G/H$ is ≤ 1. But since the operation is effective, the normal subgroup U_Z of positive dimension is not contained in H. Therefore the dimension of the orbit $U_Z H \subset G/H$ is equal to 1 and we get the morphism of algebraic operation

$$(G, G/H) \xrightarrow{(\mathrm{Id}, f)} (G, G/U_Z H) .$$

The fibre of $f: G/H \to G/U_Z H$ is $U_Z H/H$ which is a 1 dimensional homogeneous space of the commutative unipotent group U_Z, hence isomorphic to A^1. Since any effective algebraic operation on a rational curve is a suboperation of (PGL_2, P^1), there exists a morphism $(\varphi, f): (G, G/U_Z H) \to (PGL_2, P^1)$ of algebraic operations such that f is an open immersion. Let us consider the composite (φ', f') of (Id, f) and (φ, f). This gives us an exact sequence

$$1 \longrightarrow N \longrightarrow G \xrightarrow{\varphi'} PGL_2 , \qquad \text{where } N = \operatorname{Ker} \varphi' .$$

Since the fibre of f is A^1 and no simple algebraic group acts on A^1, N^0 is solvable. As we assume G is not solvable, the map φ' is surjective. Hence the sequence

$$1 \longrightarrow N \longrightarrow G \xrightarrow{\varphi'} PGL_2 \longrightarrow 1 \tag{2.5}$$

is exact, U is a subgroup of N and N is a subgroup of the automorphism group $\mathrm{Aut}_{P^1} G/H$ of A^1-bundle. Since the unipotent part of the solvable algebraic group $\mathrm{Aut}\, A^1$ is G_a, the operation of U on each fibre is abelian and hence U is abelian because the operation is effective. Therefore $U = U_Z$. The connected component N^0 is normal and we get a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & PGL_2 & \longrightarrow & 1 \\
 & & \uparrow\cup & & \| & & \uparrow & & \\
1 & \longrightarrow & N^0 & \longrightarrow & G & \longrightarrow & G/N^0 & \longrightarrow & 1 .
\end{array}$$

Since $[N: N^0] < \infty$, the morphism $G/N^0 \to PGL_2$ is finite, N^0 is the radical of G and the Lie algebra of G/N^0 is isomorphic to sl_2.

Case (C-tr-1-α). We assume $U = N^0$.

The unipotent part U is a G/U-module. Through the isogeny $SL_2 \to G/U$, U is an SL_2-module. If we consider the semi-direct product $U \cdot SL_2$, by the structure theorem in characteristic 0, there exists an isogeny $\psi: U \cdot SL_2 \to G$ such that the diagram below is commutative;

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U & \xrightarrow{i} & U\cdot SL_2 & \xrightarrow{p} & SL_2 & \longrightarrow & 1 \\
 & & \| & & \downarrow \psi & & \downarrow \text{isogeny} & & \\
0 & \longrightarrow & U & \longrightarrow & G & \xrightarrow{\varphi'} & G/U & \longrightarrow & 1
\end{array}$$

where i is the canonical inclusion, p is the projection and $SL_2 \to G/U$ is the natural isogeny. Let us put $\tilde{G} = U \cdot SL_2$ and $\tilde{H} = \psi^{-1}(H)$. Then the operation $(\tilde{G}, \tilde{G}/\tilde{H})$ is almost effective.

LEMMA (2.6). *The SL_2-module U is irreducible.*

Proof. If $U \simeq \oplus_{i=1}^{r} V_i$ be a decomposition into the direct sum of irreducible modules. Let $B \subset SL_2$ be a Borel subgroup, $C \subset B$ a Cartan subgroup, $W \subset B$ the unipotent part of B and $v_i \in V_i$ be the highest weight vector with respect to C and B. Then let us put $D = \{\lambda_1 v_1 + \cdots + \lambda_r v_r \mid \lambda_i \in k, 1 \leq i \leq r\}$. Then $D \cdot W$ is a closed abelian subgroup of dimension $r + 1$ of $\tilde{G}$. D has a 2 dimensional orbit on $\tilde{G}/\tilde{H}$ because $\lambda_j v_j$ has 1 dimensional orbit along a fibre and W has 1 dimensional orbit in horizontal direction. Hence by Lemma (1.8), $\dim D = r + 1 \leq 2$. Namely $r \leq 1$.

To determine $\tilde{H}$, we need

LEMMA (2.7). *Let L be a unipotent group (defined over an algebraically closed field characteristic 0) and M be a closed subgroup. Then M is connected.*

Proof. Induction on the dimension of L. If $\dim L = 1$, $L \simeq G_a$ and the assertion is trivial. For a general L, there exists a normal subgroup K of dimension 1 so that there exists an exact sequence

$$1 \longrightarrow K \longrightarrow L \longrightarrow L/K \longrightarrow 1 .$$

By the induction hypothesis $K \cap M$ and $\varphi(M)$ is connected, hence M is connected.

Let us now determine the closed subgroup $\tilde{H}$ of $\tilde{G}$. To this end, we fix a Borel subgroup

$$B = \left\{ \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \in SL_2 \right\}$$

and a Cartan subgroup

$$C = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in SL_2 \right\}$$

of SL_2. Let u, v be variables and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2$$

operates linearly on the 2 dimensional vector space $U_2 = ku \oplus kv$ by $u \mapsto au + cv$, $v \mapsto bu + dv$. Let U_n be the $(n-1)$-th symmetric power $S^{n-1}(U_Z)$ of U_Z. U_n is the irreducible representation of degree n of SL_2. By Lemma (2.6), there exists an integer $n \geq 1$ such that the unipotent radical U is isomorphic to U_{n+1} as an SL_2-module. Since $\dim U\tilde{H}/\tilde{H} = \dim U/U \cap \tilde{H}$ is equal to 1, $p(\tilde{H})$ is 2 dimensional hence a Borel subgroup of SL_2. By taking a conjugation if necessary, we may assume $p(\tilde{H}) = B$. Since we are in characteristic 0, $U \cap H \subset U$ is connected by Lemma (2.7). Therefore it follows, from the exact sequence

$$0 \longrightarrow U \cap \tilde{H} \longrightarrow \tilde{H} \longrightarrow B \longrightarrow 1\,, \tag{2.8}$$

that $\tilde{H}$ is connected. By exact sequence (2.8), $U \cap \tilde{H} \subset U$ is a B-invariant space with $\dim U/U \cap \tilde{H} = 1$. The B-invariant space of codimension 1 of U is uniquely determined: $U \cap \tilde{H}$ is generated by monomials $u^{n-1}v$, $u^{n-2}v$, $\cdots, v^n$ which we denote by U'_{n+1} or U'. The weights of these monomials are $n-2, n-4, \cdots, -n$. The subgroup $\tilde{H}$ is contained in $p^{-1}(B) = U\cdot B$. The image $p(\tilde{H})$ contains a torus G_m hence $\tilde{H}$ contains also a torus G_m. Taking again the conjugation by an element of $U\cdot B$, we may assume that the torus $0\cdot C \subset U\cdot B$ is in $\tilde{H}$. Let u, b, c, h be the Lie algebra of $U, B, C, \tilde{H}$. Then the Lie algebra $u + b$ of $U\cdot B$ is decomposed into the direct sum of c-eigen spaces: $u + b = \sum_{\alpha=0}^{n} W_{n-2\alpha} + (W_{-2} + c)$ where $W_i \simeq k$ and the index i is the weight of the vector space with respect to the adjoint action of c. Since $h \supset h \cap u = \sum_{\alpha=1}^{n} W_{n-2\alpha}$, $h \supset c$ and since $\dim h = \dim u + \dim b - 1$, h is also the direct sum of c-eigen spaces and h should be $\sum_{\alpha=1}^{n} W_{n-2\alpha} + c +$ (one dimensional c-eigen space) W' where W' is a subspace of $W_n + W_{-2}$. Hence $W' = W_n$ or W_{-2}. In the first case $\tilde{H} = U\cdot C$ and in the second case $\tilde{H} = (u^{n-1}v, \cdots, uv^{n-1}, v^n)\cdot B$. Since the operation $(\tilde{G}, \tilde{G}/\tilde{H})$ is almost effective, $\tilde{H}$ contains no normal subgroup of $\tilde{G}$ of positive dimension and $\tilde{H}$ can not coincide with $U\cdot C$. Hence $\tilde{H} = (u^{n-1}v, u^{n-2}v^2, \cdots, v^n)\cdot B$.

To state our conclusion we need definitions.

Definition (2.9). For an integer $n \geqslant 0$, we denote by G_n the semidirect product $U_{n+1}SL_2$, U'_{n+1} is the linear space $(u^{n-1}v, u^{n-2}v^2, \cdots, v^n)$ and H_n denotes the closed subgroup $U'_{n+1}B$ of G_n.

We have proved

Proposition (2.10). *There exists an integer $n \geqslant 0$ such that the con-*

jugacy class of the algebraic operation $(G, G/H)$ *in case* (C-tr-1-α) *is realized by* $(G_n, G_n/H_n)$.

We write explicitly the operation $(G_n, G_n/H_n)$ in terms of a local coordinate system for a later use.

PROPOSITION (2.11). *For an integer* $n \geqslant 0$, *the algebraic operation* $(G_n, G_n/H_n)$ *realizes an algebraic subgroup in* Cr_2 *consisting of the following* k*-automorphisms of* $k(x, y)$:

$$(x, y) \longmapsto \left(\frac{ax+b}{cx+d}, \frac{y+f_n(x)}{(cx+d)^n}\right), \qquad \textit{where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2,\ f_n(x) \in k[x]\ ,$$

$\deg f_n(x) \leqslant n$ *and* x, y *are independent variables over* k.

Proof. To avoid the index, we put $G_n = \tilde{G}$, $H_n = \tilde{H}$, $U_n = U$ and $U_n' = U'$.

Let

$$K = \left\{ yu^n \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in USL_2 = \tilde{G} \mid x, y \in k \right\} .$$

Then K is a closed subgroup of $\tilde{G}$ isomorphic to $G_a^{\oplus 2}$. Since $K \cap \tilde{H} = 1$, the orbit $K \cdot \tilde{H}$ in $\tilde{G}/\tilde{H}$ is isomorphic to $G_a^{\oplus 2} \simeq A^2$ and we can take x, y as a coordinate system on A^2. Let us express the regular operation of $\tilde{G}$ on $\tilde{G}/\tilde{H}$ as a rational operation on the open subset $K \cdot \tilde{H}$ of $\tilde{G}/\tilde{H}$ by using the coordinate system x, y. First let

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2 .$$

To write the operation of g in terms of coordinate system x, y is equivalent to determine x', y' such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} yu^n \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \tilde{H} = y'u^n \begin{pmatrix} 1 & x' \\ 0 & 1 \end{pmatrix} \tilde{H} . \tag{2.12}$$

Let us solve (2.12) which is equivalent to

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} yu^n \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in y'u^n \begin{pmatrix} 1 & x' \\ 0 & 1 \end{pmatrix} \tilde{H} .$$

Therefore, since we have seen $\tilde{H} = U' \cdot B$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} yu^n \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = y'u^n \begin{pmatrix} 1 & x' \\ 0 & 1 \end{pmatrix} \left(\sum_{l=1}^{n} a_l u^{n-l} v^l \right) \begin{pmatrix} a' & 0 \\ c' & a'^{-1} \end{pmatrix}$$

for suitable

$$a_l \in k\,,\quad 1 \leq l \leq n\,,\quad \begin{pmatrix} a' & 0 \\ c' & a'^{-1} \end{pmatrix} \in SL_2\,,$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} yu^n \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

$$= y'u^n \begin{pmatrix} 1 & x' \\ 0 & 1 \end{pmatrix} \Big(\sum_{l=1}^{n} a_l u^{n-l} v^l\Big) \begin{pmatrix} 1 & x' \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & x' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a' & 0 \\ c' & a'^{-1} \end{pmatrix},$$

$$y(au+cv)^n \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \Big[y'u^n + \sum_{l=1}^{n} a_l u^{n-l}(x'u+v)^l\Big] \begin{pmatrix} 1 & x' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a' & 0 \\ c' & a'^{-1} \end{pmatrix}.$$

and we get

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a' & 0 \\ c' & a'^{-1} \end{pmatrix}, \tag{2.13}$$

$$y(au+cv)^n = y'u^n + \sum_{l=1}^{n} a_l u^{n-l}(x'u+v)^l\,. \tag{2.14}$$

It follows from (2.13) $x' = (ax+b)/(cx+d)$. To solve (2.14), we notice $(au+cv)^n = ((a-cx')u + c(x'u+v))^n$. Therefore $y' = y(a-cx')^n = y/(cx+d)^n$. The operation of U on A^2 is determined in a similar way and they are the set of automorphisms $(x,y) \mapsto (x,y) + f_n(x))$, $f_n(x) \in k[x]$, $\deg f_n(x) \leq n$.

Case (C-tr-1-β). N^0 is not unipotent.

Since the fibre of f is A^1, and $\mathrm{Aut}\,A^1$ is solvable, N^0 is solvable and the dimension of a maximal torus of N^0 is at most 1 and $N^0 = U\cdot \boldsymbol{G}_m$ by Lemma (1.21). By the structure theorem, there is an isogeny $\psi\colon N^0\cdot SL_2 = \tilde{G} \to G$ such that $\psi \mid N^0$ is an isomorphism. Hence, there are an isogeny $\psi\colon U\cdot \boldsymbol{G}_m \cdot SL_2 \to G$ and a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U\boldsymbol{G}_m & \xrightarrow{\ i\ } & U\cdot \boldsymbol{G}_m \cdot SL_2 & \xrightarrow{\ p\ } & SL_2 & \longrightarrow & 1 \\ & & \cap\downarrow & & \downarrow\psi & & \downarrow & & \\ 1 & \longrightarrow & N & \longrightarrow & G & \xrightarrow{\ \varphi'\ } & PGL_2 & \longrightarrow & 1\,, \end{array}$$

where i is the canonical inclusion, p is the projection and $SL_2 \to PGL_2$ is the isogeny. The algebraic group $\varphi(U\cdot SL_2)$ is in case (C-tr-1-α). Hence U is abelian and the $\boldsymbol{G}_m \cdot SL_2$-module U is irreducible. We shall see below as G_m-module, U is of weight 1. Now let $\tilde{H} = \psi^{-1}(H)$. By (C-tr-1-α), $\tilde{H} \cap U\cdot SL_2 = (u^{n-1}v, u^{n-2}v^2, \cdots, v^n)B$. As every algebraic action of $\boldsymbol{G}_m$ on A^1 has a fixed point, $\boldsymbol{G}_m(\subset N)$ has a fixed point on the fibre $\tilde{H}/U \cap \tilde{H} \simeq A^1$. Hence we may assume that by taking the conjugation of an element of U if necessary, that the maximal torus $\boldsymbol{G}_m$ of N^0 is contained $\tilde{H}$.

LEMMA (2.15). *$\tilde{H}$ is connected.*

Proof. As $\dim N\tilde{H}/\tilde{H} = \dim N/N \cap \tilde{H} = 1$, $\dim p(\tilde{H})$ is 2 and $p(\tilde{H})$ is a Borel subgroup and in particular connected (Borel [1]).

Therefore, it is sufficient to show that $\tilde{H} \cap U\boldsymbol{G}_m$ is connected. In fact, by the exact sequence $0 \to U \to U\boldsymbol{G}_m \to \boldsymbol{G}_m \to 1$, we get $0 \to U \cap \tilde{H} \to (\tilde{H} \cap U)\boldsymbol{G}_m \to \boldsymbol{G}_m \to 1$. But, as we are in characteristic 0, the subgroup $U \cap \tilde{H}$ of the unipotent group U is connected by Lemma (2.7). Thus $\tilde{H} \cap U\boldsymbol{G}_m$ is connected.

We have shown that $\tilde{H}$ is connected and $\tilde{H} \supset U' \cdot \boldsymbol{G}_m \cdot B$, where $U' = (u^{n-1}v, u^{n-2}v, \cdots, v^n)$. But $\dim U \cdot \boldsymbol{G}_m SL_2/\tilde{H} = 2 = \dim U \cdot \boldsymbol{G}_m \cdot SL_2/U' \cdot \boldsymbol{G}_m \cdot B$. Hence $\tilde{H} = U' \cdot \boldsymbol{G}_m \cdot B$. Thus if we determine the $\boldsymbol{G}_m$-module structure U, everything is determined. Since U is an irreducible $\boldsymbol{G}_m \cdot SL_2$-module $U \simeq V \otimes U_{n+1}$ where V is an irreducible $\boldsymbol{G}_m$-module. If the representation $\boldsymbol{G}_m \to GL(U_1) \simeq \boldsymbol{G}_m$ $\lambda \mapsto \lambda^l$ is not faithful ($|l| \geq 2$), taking the same coordinate system as in the proof of Proposition (2.11), we see that the operation of $\boldsymbol{G}_m$ is given by $(x, y) \mapsto (x, \lambda^l y)$. $\lambda \in \boldsymbol{G}_m$, and is not faithful. This contradicts the assumption that $\psi | N^0$ is an isomorphism and N^0 operates faithfully. We thus proved U is a $\boldsymbol{G}_m$-module of dimension $n+1$ and of weight ± 1 hence we may assume of weight 1.

We fix notations to state the result.

DEFINITION (2.16). Let k be the $\boldsymbol{G}_m$-module of weight 1 and U_{n+1} (resp. U'_{n+1}) is the irreducible SL_2 (resp. B)-module in Definition (2.9) for $n \geqslant 0$. The irreducible $\boldsymbol{G}_m SL_2$-module $k \otimes U_{n+1}$ is also denoted by U_{n+1}. $G^{(n)}$ is the semi-direct product $U_{n+1} \cdot \boldsymbol{G}_m SL_2$ and we put $H^{(n)} = U'_{n+1} \cdot \boldsymbol{G}_m B$.

We have proved

PROPOSITION (2.17). *There exists an integer $n \geqslant 0$ such that the conjugacy class of the algebraic operation $(G, G/H)$ in case (C-tr-1-β) is realized by $(G^{(n)}, G^{(n)}/H^{(n)})$.*

We can also make explicit the operation $(G^{(n)}, G^{(n)}/H^{(n)})$ in terms of a local coordinate system.

PROPOSITION (2.18). *For an integer $n \geqslant 0$, the algebraic operation $(G^{(n)}, G^{(n)}/H^{(n)})$ realizes an algebraic subgroup in* Cr_2 *consisting of the following k-automorphisms of $k(x, y)$:*

$$(x, y) \longmapsto \left(\frac{ax+b}{cx+d}, \frac{\lambda y + f_n(x)}{(cx+d)^n}\right), \qquad \textit{where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2,\ f_n(x) \in k[x]\ ,$$

$\deg f_n(x) \leqslant n$, $\lambda \in k^*$ *and* x, y *are independent variables over* k.

Proof. Using the same coordinate system as in the proof of Proposition (2.11), the Proposition is proved in a similar way.

PROPOSITION (2.19). *For* $n \geqslant 0$, *the operation* $(G_n, G_n/H_n)$ *is a sub-operation of* $(G^{(n)}, G^{(n)}/H^{(n)})$.

Proof. In fact, consider a morphism of algebraic groups $\varphi\colon G_n = U_{n+1}SL_2 \to G^{(n)} = U_{n+1}\cdot G_m SL_2$, $\varphi(u\cdot g) = u\cdot 1\cdot g$ for $u \in U_{n+1}$, $g \in SL_2$, then $\varphi^{-1}(H^{(n)}) = H_n$ and the Proposition is proved.

Summing up Proposition (2.10), Proposition (2.17) and Proposition (2.19), we get

PROPOSITION (2.20). *Let G be a connected algebraic subgroup of the Cremona group* Cr_2 *of two variables. We assume that G is generically transitive and that G is not solvable. Let* $(G, G/H)$ *be a realization of G and U be the unipotent radical of G.*

(1) *The following are equivalent*:

(i) *The dimension of a U-orbit in* G/H *is* 1.

(ii) *The dimension of any U-orbit in* G/H *is* 1.

(2) *If one of the conditions* (i), (ii) *is satisfied, then there exists an integer* $n \geq 0$ *such that in* Cr_2 *the conjugacy class of G is contained in the conjugacy class of* $(G^{(n)}, G^{(n)}/H^{(n)})$.

Proof. Let us prove the assertion (1). It is sufficient to show (i) $\Rightarrow$ (ii). For $g \in G$, $UgH = gg^{-1}UgH = gUH$ and the U-orbit gH is the translation by g of the U-orbit UH. The assertion (2) is a consequence of Proposition (2.10), Proposition (2.17) and Proposition (2.19).

We shall see the conjugacy class of $(G^{(n)}, G^{(n)}/H^{(n)})$ is maximal in Cr_2 for $n \geqslant 2$ (Theorem (2.25)) but not so for $n = 0, 1$ (Proposition (2.21) and Proposition (2.24)).

PROPOSITION (2.21). *In the Cremona group* Cr_2 *of two variables, the conjugacy class determined by the almost effective realization* $(G^{(1)}, G^{(1)}/H^{(1)})$ *is a subgroup of the conjugacy class determined by an effective realization* $(PGL_3, \boldsymbol{P}_2)$.

Proof. Let $p = \{(a_{ij}) \in SL_3 \mid a_{1j} = 0, j = 2, 3\}$. P is a parabolic subgroup and $(SL_3, SL_3/P)$ is an almost effective realization of the conjugacy class of $(PGL_3, \boldsymbol{P}^2)$. Let $\varphi\colon G^{(1)} = U_{(2)}\cdot G_m\cdot SL_2 \to SL_3$ be defined by the formula,

$$\varphi((au+bv)\cdot t\cdot A)=\begin{pmatrix} & & a\\ A & & b\\ 0 & 0 & t^{-1}\end{pmatrix}\qquad \text{for } a, b\in k,\ t\in k^*,\ A\in SL_2\,.$$

Then $\varphi^{-1}(p)=U'_{(1)}G_m\cdot B=H^{(1)}$ and φ defines a morphism $(\varphi, f)\colon(G^{(1)}, G^{(1)}/H^{(1)})\to(SL_3, SL_3/P)$ of algebraic operations. Since $\varphi(G^{(1)})$ has an open orbit on P^2 and φ is injective, f is birational. The Proposition now follows from Proposition (1.10).

Case (C-tr-2). G is generically transitive and reductive.

Let R be the radical of G. Since $k=C$, there exists a semi-simple algebraic group G' and an isogeny $\varphi\colon R\times G'\to G$ such that the restriction of φ on R is an isomorphism. Let $(G, G/H)$ be an effective realization. By corollaire 1, p. 521, Demazure [3], the rank of $G\leq 2$. Since G_2 contains no Lie subalgebra of codimension 2, we can exclude the simple algebraic group of which the Lie algebra is G_2. Hence there exist an algebraic group $\tilde{G}$ and an isogeny $\psi\colon\tilde{G}\to G$ such that ψ is an isomorphism on the radical of $\tilde{G}$ and $\tilde{G}$ is isomorphic to one of the following: (i) SL_3, (ii) $SL_2\times SL_2$, (iii) $G_m\times SL_2$, (iv) SL_2, (v) $G_m\times G_m$. Since $(G, G/H)$ is of de Jonquières type, by Lemma (2.1) there exists a non-trivial morphism of law chunks of algebraic operations $(\varphi, f)\colon(G, G/H)\to(PGL_2, P^1)$, hence the first group SL_3 is excluded. Let us determine all algebraic subgroup $\tilde{H}\subset\tilde{G}$ of codimension 2 such that

($*$) the radical operates on $(\tilde{G}/\tilde{H})$ effectively and the algebraic operation $(\tilde{G}, \tilde{G}/\tilde{H})$ is almost effective and of de Jonquières type. This gives us almost effective realizations of all the group G in case (C-tr-2).

LEMMA (2.22). *Up to isomorphism, we have a list of all the algebraic subgroup of* $\tilde{G}$ *of codimension* 2 *satisfying the condition* (*):

group $\tilde{G}$	subgroups $\tilde{H}$
$SL_2\times SL_2$	$B\times B$, B is a Borel subgroup of SL_2
$G_m\times SL_2$	$\left\{t^n\times\begin{pmatrix} t & 0\\ x & t\end{pmatrix}\,\middle\vert\, x\in k,\ t\in k^*\right\}$, n is a non-negative integer
SL_2	$U_n=\left\{\begin{pmatrix} s & 0\\ x & s^{-1}\end{pmatrix}\,\middle\vert\, x\in k,\ s^n=1\right\}$, n is a non-negative integer

$$H = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \middle| t \in k^* \right\}$$

$$D_\infty = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^m \middle| m = 0, 1,\ t \in k^* \right\}$$

$\boldsymbol{G}_m \times \boldsymbol{G}_m$ 1 .

Proof of the Lemma. If $\tilde{G} = SL_2 \times SL_2$, we show $\tilde{H}$ is solvable. If $\tilde{H}$ is not solvable, then there is a non-trivial morphism $SL_2 \to \tilde{H}$. Hence, there is a non-trivial morphism $SL_2 \to SL_2 \times SL_2$. This morphism is, up to automorphism of SL_2, one of the following: (i) $x \mapsto (x, 1)$, (ii) $x \mapsto (1, x)$, (iii) $x \mapsto (x, x)$. The morphisms (i) and (ii) do not occur. For $\tilde{H}$ does not contain any normal algebraic subgroup of positive dimension because $(\tilde{G}, \tilde{G}/\tilde{H})$ is almost effective. Let Δ denote the morphism (iii). The Lie algebra $sl_2 \times sl_2$ is considered as an sl_2-module by Δ_*. Then, as sl_2-module, $sl_2 \times sl_2$ is isomorphic to $sl_2 \oplus sl_2$. Hence $\Delta_* sl_2$ is a maximal sl_2-invariant space and in particular maximal Lie algebra of $sl_2 \times sl_2$. This proves there is no algebraic subgroup of $SL_2 \times SL_2$ containing $\Delta(SL_2)$ and of codimension 2. Now we may assume $\tilde{H}$ is solvable. Therefore the connected component $\tilde{H}^0$ is contained in a Borel subgroup $B \times B$. Counting the dimension, we conclude $\tilde{H}^0 = B \times B$. Since any parabolic subgroup is connected, $\tilde{H} = B \times B$ (Borel [1]).

If $G = \boldsymbol{G}_m \times SL_2$, $\tilde{H}$ is solvable. For, otherwise, $\tilde{H}$ contains a normal subgroup of positive dimension SL_2. As $\boldsymbol{G}_m$ operates on $\tilde{G}/\tilde{H}$ effectively and $\boldsymbol{G}_m$ is in the center of $\tilde{G}$, $\boldsymbol{G}_m \cap \tilde{H} = \{1\}$ and we get an exact sequence

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \boldsymbol{G}_m & \longrightarrow & \boldsymbol{G}_m \cdot SL_2 & \xrightarrow{p} & SL_2 & \longrightarrow & 1\,, \\ & & & & \cup & & \cup & & \\ & & & & \tilde{H} & \xrightarrow{\sim} & B & & \end{array}$$

where p is the projection. Since the dimension of the image $p(\tilde{H})$ is 2, $p(\tilde{H})$ is a Borel subgroup B of SL_2. We may assume $B =$ lower triangular matrices in SL_2. Now we conclude

$$\tilde{H} = \left\{ t^m \times \begin{pmatrix} t & 0 \\ x & t \end{pmatrix} \middle| t \in k^*,\ x \in k \right\}$$

for an appropriate integer m. Since m and $-m$ give the same subgroup, we may assume m is non-negative.

If $\tilde{G} = SL_2$, the result is well known. If $\tilde{G} = \boldsymbol{G}_m \times \boldsymbol{G}_m$, the assertion is trivial.

PROPOSITION (2.23). *The notation being as in Lemma* (2.22), *the conjugacy class in* Cr_2 *determined by an almost effective realization* $(G, G/H)$ *will be simply called the conjugacy class of* $(G, G/H)$.

(1) *The conjugacy class of*

$$\left(\boldsymbol{G}_m \times SL_2,\ \boldsymbol{G}_m \times SL_2 \Big/ \left\{t^n \times \begin{pmatrix} t & 0 \\ x & t \end{pmatrix} \Big| t \in k,\ t \in k^*\right\}\right)$$

is a subgroup of the conjugacy class of $(G^{(n)}, G^{(n)}/H^{(n)})$, *for* $n \geq 0$. (*Definition* (1.9)).

(2) *The conjugacy class of* $(SL_2, SL_2/U_n)$ *is a subgroup of the conjugacy class of* $(G^{(n)}, G^{(n)}/H^{(n)})$, *for* $n \geq 0$.

(3) *The conjugacy class of* $(SL_2, SL_2/T)$ *is a subgroup of the conjugacy class of* $(SL_2 \times SL_2,\ SL_2 \times SL_2/B \times B)$.

(4) *The conjugacy class of* $(SL_2, SL_2/D_\infty)$ *is a subgroup of the conjugacy class of* $(PGL_3, \boldsymbol{P}_2)$.

(5) *The conjugacy class of* $(\boldsymbol{G}_m \times \boldsymbol{G}_m,\ \boldsymbol{G}_m \times \boldsymbol{G}_m/1)$ *is a subgroup of the conjugacy class of* $(PGL_3, \boldsymbol{P}_2)$.

Proof. To prove the first assertion, by Proposition (1.10) it is sufficient to construct a morphism of algebraic operations

$$(\varphi, f)\colon \left(\boldsymbol{G}_m \times SL_2,\ \boldsymbol{G}_m \times SL_2 \Big/ \left\{t^n \times \begin{pmatrix} t & 0 \\ x & t^{-1} \end{pmatrix} \Big| x \in k,\ t \in k^*\right\}\right) \longrightarrow (G^{(n)}, G^{(n)}/H^{(n)})$$

such that f is a birational morphism. Let $\alpha\colon G^{(n)} = U_{(n+1)} \cdot \boldsymbol{G}_m \cdot SL_2 \to G^{(n)} = U_{(n+1)} \cdot \boldsymbol{G}_m \cdot SL_2$ be the inner automorphism $\alpha(x) = v^n \cdot x \cdot (-v^n)$ for $x \in G^{(n)}$. An easy calculation shows that

$$\alpha(0 \cdot \boldsymbol{G}_m \cdot SL_2) \cap H^{(n)} = \left\{0 \cdot t^n \cdot \begin{pmatrix} t & 0 \\ x & t^{-1} \end{pmatrix} \in \boldsymbol{G}_m \cdot U \cdot SL_2 \,|\, x \in k,\ t \in k^*\right\}.$$

If we consider the composite map

$$\boldsymbol{G}_m \times SL_2 \xrightarrow{\ i\ } U_{(n+1)} \cdot G_m \cdot SL_2 \xrightarrow{\ \alpha\ } U_{n+1} \cdot G_m \cdot SL_2$$

where $i((t, A)) = 0 \cdot t \cdot A$ for

$$t \in \boldsymbol{G}_m\,, \quad A \in SL_2\,, \quad (\alpha \cdot i)^{-1} H^{(n)} = \left\{t^n \times \begin{pmatrix} t & 0 \\ x & t^{-1} \end{pmatrix} \in \boldsymbol{G}_m \times SL_2\right\}$$

and the first assertion is proved. As for the second assertion, by the inclusion $SL_2 \hookrightarrow G_m \times SL_2$ $A \mapsto (1, A)$, $(SL_2, SL_2/U_n)$ is a suboperation of

$$\left(G_m \times SL_2,\ G_m \times SL_2 \Big/ \left\{t^n \times \begin{pmatrix} t & 0 \\ x & t^{-1} \end{pmatrix}\right\}\right)$$

and we have shown (1). Therefore the assertion (2) is proved. To prove the assertion (3), we give only φ of morphism (φ, f) because f is uniquely determined by f. $\varphi: SL_2 \to SL_2 \times SL_2$ is defined by $\varphi(A) = (A, {}^tA^{-1})$. We have $\varphi^{-1}(B \times B) = H$. The assertion (4) is proved in Proposition (2.4). The assertion (5) is trivial.

As the conjugacy class realized by the algebraic operation $(G^{(1)}, G^{(1)}/H^{(1)})$ failed to be maximal (Proposition (2.21)), $(G^{(0)}, G^{(0)}/H^{(0)})$ is not maximal either.

PROPOSITION (2.24). *The conjugacy class determined by $(G^{(0)}, G^{(0)}/H^{(0)})$ is a subgroup of the conjugacy class realized by the almost effective algebraic operation $(SL_2 \times SL_2, SL_2/B \times SL_2/B)$.*

Proof. By abuse of notation, let B be the Borel subgroup of PSL_2 consisting of the matrices $(a_{ij})_{1 \le i, j \le 2} \in PGL_2$, $a_{12} = 0$. Then $(PGL_2 \times PGL_2, PGL_2/B \times PGL_2/B)$ realizes the same conjugacy class as $(SL_2 \times SL_2, SL_2/B \times SL_2/B)$. It is sufficient, by Proposition (1.3) and Proposition (1.10), the construct a morphism

$$(\varphi f): (G^{(0)}, G^{(0)}/H^{(0)}) \longrightarrow (PGL_2 \times PGL_2, PGL_2/B \times PGL_2/B)$$

of algebraic operations such that f is birational. Since both spaces are homogeneous, f is uniquely determined by φ. Let $\varphi: G^{(0)} = U_1 \cdot G_m \cdot SL_2 \to SL_2 \times SL_2$ be defined by

$$\varphi(w, t, A) = \left(\begin{pmatrix} t & w \\ 1 & 0 \end{pmatrix}, A\right)$$

for $w \in U_1$, $t \in G_m$, $A \in SL_2$. Then φ is a morphism of algebraic groups and $\varphi^{-1}(B \times B) = 0 \cdot 1 \cdot B$, hence φ gives an open immersion f.

THEOREM (Enriques) (2.25). *Let G be a connected algebraic subgroup of the Cremona group Cr_2 of two variables.*

(1) *The conjugacy class of G in Cr_2 is a subgroup of the conjugacy class realized by one of the following almost effective algebraic operations*

(i) (PGL_3, P_2),

(ii) $(G^{(n)}, G^{(n)}/H^{(n)})$, $n = 2, 3, \cdots$

(iii) $(PGL_2 \times PGL_2, PGL_2 \times PGL_2/B \times B)$, *B is a Borel subgroup of* PGL_2.

(2) *There exists no inclusions among the conjugacy classes realized by the almost effective algebraic operations* (i), (ii), (iii).

Proof. The second assertion follows from Proposition (1.10) if we notice that there is no inclusion among the Lie algebras of the conjugacy class of the almost effective realizations (i), (ii), (iii). The first assertion is already verified under several additional assumptions on G:

(1) G is primitive (A);

(2) G is imprimitive but not of de Jonquières type ((B), Corollary (2.3), Proposition (2.4));

(3) G is of de Jonquières type, generically transitive and neither solvable nor reductive ((C-tr-1), Proposition (2.20), Proposition (2.21), Proposition (2.24);

(4) G is of de Jonquières type, generically transitive and reductive ((C-tr-2), Lemma (2.22) Proposition (2.23)).

The rest of the paper is devoted to the proof of the Theorem. The proof will be done as follows:

(5) G is of de Jonquières type, generically transitive and solvable (Proposition (2.26));

(6) G is generically intransitive (C-intr).

Case (C-tr-4). G is generically transitive and solvable.

In this case Theorem (2.25) follows from the following Proposition.

PROPOSITION (2.26). *Let G be a (connected) algebraic subgroup of the Cremona group* Cr_2 *of* 2 *variables. If G is generically transitive and solvable, G is contained in one of the conjugacy classes of the almost effective realizations* (i), (ii), (iii) *of Theorem* (2.25).

Proof. By Corollaire 1, p. 521, Demazure [3], we may assume G is not a torus. Let U denote the unipotent part of G and U_Z its center. If U_Z has 2 dimensional orbit, then by Lemma (1.12) G is a suboperation of affine transformation and hence a suboperation of (PGL_2, P_2). Thus we may assume the orbits of U_Z are 1 dimensional as in Proposition (2.20). Now Proposition (2.26) follows from Proposition (2.21), Proposition (2.24) and from

LEMMA (2.27). *Under the same assumption as in Proposition* (2.26), *if* *the dimension of the U_Z-orbits are* 1, *G is a subgroup of the almost effective realization* $(G^{(n)}, G^{(n)}/H^{(n)})$ *for an appropriate integer* $n \geqslant 0$.

Proof of the Lemma. Let $(G, G/H)$ be an effective realization of G. The morphism of algebraic operations $(G, G/H) \to (G, G/U_Z H)$, gives an exact sequence

$$1 \longrightarrow N \longrightarrow G \xrightarrow{\pi} G' \longrightarrow 1 . \tag{2.28}$$

G' is a solvable subgroup of PGL_2. Hence, G' is isomorphic to $\boldsymbol{G}_a, \boldsymbol{G}_m$ or the affine transformation group GTA_1 of 1 variable. The proof of Lemma (2.27) is given under the hypotheses $G' \simeq$ affine transformation group GTA_1. The proof in other cases is similar and simpler, hence omitted. Therefore from now on we assume $G' \simeq GTA_1$. Let U be the unipotent part of the solvable group G and T a (fixed) maximal torus. The unipotent part N_U of N is abelian for the same reason as in case (C-tr-1). The maximal torus T acts on U and on its center U_Z by inner automorphism. Since N_U is normal in U, N_U is a T-module. Since by Lemma (2.7) we have $N_U = U \cap N$, restricting exact sequence (2.28) on U, we get an exact sequence

$$0 \longrightarrow N_U \longrightarrow U \xrightarrow{\pi} \boldsymbol{G}_a \longrightarrow 0 \tag{2.29}$$

of T-groups.

SUBLEMMA (2.30). *Let S be a torus and U_1 be a unipotent group. We assume S acts on U_1. Let U_2 be a S-invariant closed normal subgroup of U_2. Let U_3 be the quotient S-group U_1/U_2;*

$$1 \longrightarrow U_2 \longrightarrow U_1 \xrightarrow{\pi} U_1/U_2 \longrightarrow 1 . \tag{2.31}$$

If U_1/U_2 is 1 *dimensional (hence U_1/U_2 is isomorphic to $\boldsymbol{G}_a$), then exact sequence* (2.31) *of S-groups, S-splits. Namely, there exists an S-morphism s; $U_1/U_2 \to U$ such that $\pi \circ s =$* Id.

Proof. Let Z be the center of U_1. We prove the existence of S-section s by induction on the dimension of U_1. Before we begin the induction, we notice that if U_1 is abelian, the Sublemma follows from the semi-simplicity of S-module U_1. In particular, if the restriction $\pi | Z$ is surjective, then an S-section exists. If $\dim U_1 = 1$, then lemma is obvious. If $\pi | Z \colon Z \to U_1/U_2$ is surjective, then the an S-section exists as we noticed above. If $\pi | Z$ is not surjective, then $\pi(Z) = 0$ and we get a commutative diagram of S-groups,

$$\begin{tikzcd}
0 \arrow[d] & \\
Z \arrow[d] & \\
U_1 \arrow[r, "\pi"] \arrow[d, "p"'] & U_1/U_2 \\
U_1/Z \arrow[ur] \arrow[d] & \\
1 &
\end{tikzcd}$$

Since $\dim U/Z < \dim U_1$, by induction hypothesis there exists an S-equivariant section s of $U_1/Z \to U_1/U_2$. If we restrict the vertical exact sequence on $s(U_1/U_2)$, an S-equivariant section of p over $s(U_1/U_2)$ exists by the same arguments as in the proof of Lemma (1.14).

Let us come back to the proof of Lemma (2.27). By sublemma (2.30), there exists a T-section s of exact sequence (2.29). $s(G_a)T$ is a closed subgroup of G. Since $s(G_a)T$ is solvable, there exists a 1 dimensional $s(G_a)T$-invariant space V of $s(G_a)T$-module U_Z. We put $W = V \cdot s(G_a)$. Since 1 dimensional representation V of G_a is trivial, W is a closed subgroup of G isomorphic to $G_a \times G_a$. The orbit $W \cdot H$ in G/H coincides with G/H. To see this, let us consider a commutative diagram;

$$\begin{tikzcd}
W \cdot H \arrow[d] & \subset & G/H \arrow[d] \\
W \cdot U_Z H & \subset & G/U_Z H
\end{tikzcd}$$

and we observe that $W \cdot U_Z H = G/U_Z H$ and that the both fibrations have the same fibres. In fact, $W \cdot U_Z H =$ the open orbit of $G_a (\subset G')$ in $A^1 = G/U_Z H$ and $W \cdot U_Z H = G/U_Z H$. As for fibres, $A^1 \simeq V \cdot H \subset U_Z H \simeq A^1$ hence $V \cdot H = U_Z \cdot H$. Since W has a 2 dimensional orbit and W is abelian, W is canonically isomorphic to the orbit $W \cdot H$. Let us fix an isomorphism $G_a \times G_a \simeq W$ and a coordinate system on G_a. Then they define a coordinate system (y, x) on W hence on $W \cdot H = G/H$. We use this coordinate system (y, x) to write explicitly the operation $(G, G/H)$. In order to show G is a subgroup of the conjugacy class of the almost effective realization $(G^{(n)}, G^{(n)}/H^{(n)})$ for an appropriate integer $n \geq 0$, it is sufficient to prove the operations of the maximal torus T and the unipotent part U are contained in the group of the birational automorphisms in Proposition (2.18), when they are expressed by using the above coordinate system on G/H.

As in the proof of Lemma (1.12), it is easy to see that the operation of the maximal torus T is a suboperation of the operation of 2 dimensional torus; $(y, x) \to (t_1^a y, t_2^b x)$. Since U is the semi-direct product $N_U \cdot s(\boldsymbol{G}_a)$, we have to write the operations of $N_U = U \cap N$ and $s(\boldsymbol{G}_a)$ using the coordinate system (y, x). The operation of $s(\boldsymbol{G}_a)$ is $(y, x) \to (y, x + c)$ for $c \in k$. As we have seen above N_U is abelian and is a direct sum $N_U = (N_U \cap H) \cdot V$. The operation of V is $(y, x) \to (y + d, x)$ for $d \in k$. Therefore it suffices to write the operation of $N_U \cap H$. To describe the operation of $N_U \cap H$, we need some notations. Let us take a base $e_1, \cdots, e_l$ of N_U such that the $s(\boldsymbol{G}_a)$-module structure on N_U is given by, for $x \in \boldsymbol{G}_a$,

$$(xe_1, xe_2, \cdots, xe_l) = (e_1, e_2, \cdots, e_l)(f_{ij}(x)) ,$$

where $f_{ij}(x)$ $1 \le i, j \le l$ are polynomials in x and the matrix $F(z) = (f_{ij}(x))$ is lower triangular ($f_{ij}(x) = 0$ if $1 \le i < j \le l$) and $f_{ii}(x) = 1$ for $1 \le i \le l$ (Theorem (Kolchin) LA 5.7, Serre [11]). By Lemma (1.8), $s(\boldsymbol{G}_a)$-invariant subspace of N_U is 1 dimensional. Therefore $\{\lambda e_l \mid \lambda \in k\} = V$. Let $\sum_{i=1}^n a_i e_i = w \in N_U$. To write the operation of w using the coordinate system (y, x) is equivalent to determining (y', x') such that

$$\left(\sum_{i=1}^{l} a_i e_i\right)(y e_l \cdot x) = (y' e_l \cdot x')\left(\sum_{i=1}^{l} b_i e_i\right) \qquad \text{in } N_U \cdot s(\boldsymbol{G}_a) \tag{2.32}$$

where x, x' are considered as elements of $s(\boldsymbol{G}_a)$ and $\sum_{i=1}^l b_i e_i \in N_U \cap H$. From (2.32), we get

$$\left(\sum_{i=1}^{l} a_i e_i + y e_l\right) \cdot x = \left(y' e_l + x'\left(\sum_{i=1}^{l} b_i e_i\right) x'^{-1}\right) \cdot x' .$$

Hence $x = x'$ and

$$\sum_{i=1}^{l} a_i e_i + y e_l = y' e_l + \sum_{i=1}^{l} \sum_{k=1}^{l} b_i f_{ki}(x) e_k . \tag{2.33}$$

SUBLEMMA (2.34). *There exist $\lambda_i \in k$, $1 \le i \le l - 1$ such that, for any $\sum_{i=1}^l a_i e_i \in N_U \cap H$, the coefficient a_l is equal to a linear combination $\sum_{i=1}^{l-1} \lambda_i a_i$.*

Proof. $N_U \cap H$ is a linear subspace of N_U of codimension 1 and since the operation of $N_U \cdot s(G_a)$ is transitive and effective, $N_U \cap H$ contains no $s(G_a)$-invariant subspace of positive dimension. Hence $(N_U \cap H) \cap V = 0$. Let p (resp. q): $N_U = ke_1 \oplus ke_2 \oplus \cdots \oplus ke_l \to ke_1 \oplus ke_2 \oplus \cdots \oplus ke_{l-1}$ (resp. ke_l) be the projection. The restriction $p_l | N_U \cap H$ is an isomorphism because $p | N_U \cap H$ is injective and because $\dim N_U \cap H = l - 1$. Thus, $q \circ (p | N_U \cap H)^{-1}$ is a homomorphism and the sublemma is proved. q.e.d.

From (2.33), we can determine

$$a_1 = b_1 f_{11}(x) = b_1 , \qquad a_2 = b_1 f_{21}(x) + b_2 f_{22}(x) = a_1 f_{21}(x) + b_2$$

and inductively b_i, $1 \leq i \leq l - 1$, hence also b_l by sublemma (2.34), are expressed as linear combinations of a_i's whose coefficients are polynomials in x. Therefore $y' = y +$ linear combination of a_i's whose coefficients are polynomials in x. The operation of $w = \sum_{i=1}^{l} a_i e_i \in N_U$ on $W \cdot H \subset G/H$ is written in terms of coordinate system (y, x); $(y, x) \mapsto (y', x') - (y +$ linear combination of a_i's whose coefficients are polynomial in $x, x)$. Therefore, for an appropriate n the operations of T, $s(G_a)$, V, N_U (hence of $G = ((N_U, V)s(G_a))T)$ is contained in the k-automorphism group in Proposition (2.18) and Lemma (2.27) is proved.

Case (C-intr). G is generically intransitive.

Let (G, X) be a realization of G. Then by Theorem 2, p. 407, Rosenlicht [7], there exists a morphism $(\varphi, f): (G, X) \to (1, V)$ of law chunks of algebraic operation such that $\dim V = 1$ and f is dominant. By the Lüroth theorem, V is rational. Replacing X by an open subset of X, we may assume (φ, f) is a morphism of algebraic operations and V is non-singular.

This case is divided into two subcases: (i) G is reductive, (ii) G is not reductive. We shall see in the second subcase G is necessarily solvable.

Bubcase (C-intr-1). G is generically intransitive and reductive.

Since the fibre of f is 1 dimensional, it follows from our assumption and Lemma (1.22) that G is a algebraic subgroup of PGL_2. Hence G is either G_m or PGL_2. If $G = G_m$, then by Corollaire 1, p. 521, Demazure [3], (G_m, X) is a suboperation of (PGL_3, P^2). Thus, we assume $G = PGL_2$. Then $f: X \to V$ is P^1-bundle. Since the Brauer group of a curve vanishes (cf. Grothendieck [5]), $f: X \to V$ is a locally trivial P^1 = bundle. Replacing V by an open subset V' and X by $f^{-1}(V')$, we may assume f is trivial, namely $X \simeq V \times P^1$. Let $s: V \to X$ be a section and we choose a Borel subgroup B of PGL_2. Let us recall the following well known facts valid for any semi-simple group and its Borel subgroup.

(2.35.1) Any Borel subgroup of PGL_2 is conjugate to the Borel group B and the normalizer of B coincides with B itself. This gives a 1: 1 correspondence

$$PGL_2/B \xrightarrow{\Psi} \{\text{Borel subgroups of } PGL_2\} \qquad (gB \longrightarrow gBg^{-1}) .$$

(2.35.2) The fibering given by the projection $p: PGL_2 \to PGL_2/B$ is locally trivial for the Zariski topology.

Let us now consider the map F: $V \to PGL_2/B$ defined as follows; $v \in V$, $v \mapsto \Psi^{-1}$ (stabilizer at $s(v)$). By (2.35.2), replacing V by an open subset, we may assume there exist a morphism $\tilde{F}$: $V \mapsto PGL_2$ such that the diagram

$$\begin{tikzcd} & PGL_2 \arrow[d, "p"] \\ V \arrow[ur, "\tilde{F}"] \arrow[r, "F"] & PGL_2/B \end{tikzcd}$$

is commutative. Let $(1 \times PGL_2, V \times PGL_2/B) = (PGL_2, V \times PGL_2/B)$ be the product operation. Let f: $V \times PGL_2/B \to X$ be a morphism defined by $h((z, gB)) = g\tilde{F}(z)^{-1}s(z)$. The morphism h is well-defined and is an isomorphism. Then (Id, h): $(PGL_2, V \times PGL_2/B) \to (PGL_2, X)$ is an isomorphism of algebraic operations. Hence (G, X) is considered as a suboperation of the operation (iii) of Theorem (2.25).

Subcase (C-intr-2). G is generically intransitive and not reductive.

Let U be the unipotent radical of G. By Theorem 2, p. 407, Rosenlicht [7] we may assume there exist a non-singular rational curve V and a morphism of algebraic operations $(1, f)$: $(U, X) \to (1, V)$ such that the fibres of f are U-orbits and f is faithfully flat. Hence the fibres of f are isomorphic to A^1. Since U is normal, the operation of G respects the fibration f: $X \to V$. As the dimension of G-orbits 1, G operates trivially on V. Therefore we get a morphism of algebraic operations $(1, f)$: $(G, X) \to (1, V)$. Let G' be a semi-simple, closed, connected subgroup of G. Then, by Lemma (1.21) G' is a subgroup of Aut A^1 which is solvable. Hence $G' = 1$ and G is solvable. By Lemma (1.21), the rank of $G \leq 1$. Since the unipotent part U of respects the fibration f: $X \to V$ whose fibre is A^1, the operation of U on each fibre of f is abelian and hence U is abelian. Consequently,

U is isomorphic to $G_a^{\oplus r}$ for a certain integer r. By Theorem 10, p. 426, Rosenlicht [9], we may assume f is trivial i.e., $X = V \times A^1$. Taking V smaller, we may assume V is non-singular and affine. To define an operation of $G_a^{\oplus r}$ on X respecting the fibering f is equivalent to giving a morphism of group schemes over V, $V \times G_a^{\oplus r} \to \mathrm{Aut}_V\, V \times A^1 = V \times \mathrm{Aut}\, A^1$ and hence a morphism of group schemes over V, $V \times G_a^{\oplus r} \to V \times G_a$. Therefore, there is a 1:1 correspondence:

$$\{\text{operations of } G_a^{\oplus r} \text{ on } X \text{ respecting the fibrations } f\} \xleftrightarrow{1:1}$$
$$\uparrow$$
$$\mathrm{Hom}_{V\text{-group scheme}}(V \times G_a^{\oplus r},\ V \times G_a) \simeq H^0(V, O_v)^{\oplus r}\ .$$

Hence there exist $f_1, f_2, \cdots, f_r \in H^0(V, O_v)$ such that the operation of $G_a^{\oplus r} = U$ on $V \times A^1$ is given by $(x, y) \mapsto (x, y + \sum_{i=1}^r a_i f_i)$ for $(a_1, a_2, \cdots, a_r) \in G_a^{\oplus r}$. To normalized this operation, let $\overline{V}$ be the non-singular compactification of V so that $\overline{V} \simeq P^1$. Let $\{P_1, P_2, \cdots, P_s\} = \overline{V} - V$. As a local coordinate on V, we take a global coordinate on $A^1 = \overline{V} - P_s$. Let $g \in H^0(V, O_V)$ such that g^{-1} is also regular on V and such that $gf_i \in H^0(\overline{V} - P_s, O_{\overline{V}-P_s})$ for $1 \le i \le r$. If we twist the coordinate y by $(x, y) \mapsto (x, g(x)u)$, the operation of $U = G_a^{\oplus r}$ becomes $(x, y) \to (x, y + \sum_{i=1}^r a_i g(x) f_i x)$ and hence U is a suboperation of Proposition (2.20) for suitable n. It remains to control the operation of $G_m (\subset G)$ on the fibre. To define a G_m operation on $V \times A^1$ respecting the fibration is equivalent to giving a morphism of group schemes over V, $V \times G_m \to \mathrm{Aut}_V\, V \times A^1 = V \times \mathrm{Aut}\, A^1$. Since any 1 dimensional torus in $\mathrm{Aut}\, A^1$ is conjugate to a fixed 1 dimensional torus, to define a G_m operation on $V \times A^1$ is equivalent to give a morphism $V \to \mathrm{Aut}\, A^1/\text{Normalizer of } G_m = \mathrm{Aut}\, A^1/G_m \simeq A^1$. As $\mathrm{Aut}\, A^1 \to \mathrm{Aut}\, A^1/G_m$ has a section, we may assume as in (C-intr-1). If we change the coordinate by $(x, y) \mapsto (x, y + f(x))$ for a certain $f \in H^0(V, O_V)$, the $G_m(\subset G)$ operation is given by $(x, y) \mapsto (x, ty)$ for $t \in G_m$. Since the coordinate change $(x, y) \mapsto (x, y + f(x))$ does not affect the operation of $U = G^{\oplus r}$, G is a suboperation of Proposition (2.20) for a suitable n. This completes the classification in Case (C-intr), hence the proof of Theorem (2.25).

As a concluding remark, let us notice the maximal connected algebraic groups in Cr_2 are related to the relatively minimal models of the rational surfaces. The homogeneous space $G^{(n)}/H^{(n)}$ is an A^1-bundle over $G^{(n)}/U_{n+1}H^{(n)} = P^1$. Since the SL_2-orbit $SL_2 \cdot U_{n+1}H^{(n)} = SL_2/B = P^1$, the A^1-bundle $G^{(n)}/H^{(n)} \to G^{(n)}/U_{n+1}H^{(n)} = P^1$ has a section hence the A^1-bundle is a line bundle. It is easy to see $G^{(n)}/H^{(n)}$ is isomorphic to $V(\mathcal{O}_{P^1}(n))$ over $\boldsymbol{P}^1$ and $(G^{(n)}, V(\mathcal{O}_{P^1}(n)))$ is equivariantly completed to $(\mathrm{Aut}^0 F_n, F_n)$ where $F_n = P(\mathcal{O}_{P^1} \oplus \mathcal{O}_{P^1}(n))$. Since F_n is projective, $\mathrm{Aut}^0 F_n$ is an algebraic group. By the maximality of $G^{(n)}$ in Cr_2, $G^{(n)} = \mathrm{Aut}^0 F_n$. The rational surfaces $\boldsymbol{P}^2$, $F_n(n \geqslant 2)$, $PGL_2/B \times PGL_2/B = \boldsymbol{P}^1 \times \boldsymbol{P}^1$ are exactly relatively minimal models of rational surfaces. We have proved

THEOREM (2.36). *The maximal connected algebraic groups in* Cr_2 *are exactly the connected components of* 1 *of the automorphism groups of the relatively minimal models of rational surfaces.*

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Nagoya University