SYMPLECTIC AUTOMORPHISMS OF $\mathbf{CP}^2$ AND THE THOMPSON GROUP T

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1. Introduction

Study of the group of birational automorphisms of $\mathbf{CP}^2$ preserving the meromorphic 2-form $\frac{dx \wedge dy}{xy}$ leads very fast to the understanding of the presence of piecewise linear geometry behind and also of deep relations with cluster algebras. Actually, there exists as explained in the paper a morphism from the group $Symp$ of such birational automorphisms to the group of piecewise linear automorphisms of $\mathbf{Z}^2$, which is a famous Thompson group T. One particular simple automorphism $P : (x, y) \mapsto (y, \frac{1+y}{x})$ looks also simply in piecewise linear world and gives rise to comparisons with mutations. So we study the subgroup of $Symp$ generated by this P and $SL(2, \mathbf{Z})$. The main result of this paper is the construction of a linear representation of this subgroup, the space where it acts is the inductive limit of Picard groups of the projective system of surfaces, on which this subgroup acts by automorphisms. This linear representation is very similar to the representation of the mapping class group of some cluster algebra, for which all the seeds are the same and the cluster variables of a seed are parameterized by $\mathbf{Z}^2$.

For our needs we have to use the Thompson group T, so in the Appendix we find another representation of it in terms of generators and relations, which seems to be new and better adopted for a piecewise linear interpretation.

The paper is organized in the following way: first we construct a morphism from the group $Symp$ of symplectic automorphisms of $\mathbf{CP}^2$ to the Thompson group T, then we gather some information about the later, next we find geometrically presentation of H, which is some natural subgroup of $Symp$. Appendix contains a presentation of T in terms of elements, natural in our context.

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1.1. **Notations.** X,Y,Z homogeneous coordinates on $\mathbf{CP}^2$, we will mainly use affine coordinates $x = \frac{X}{Z}$, $y = \frac{Y}{Z}$.

$Symp$ - group of birational automorphisms of $\mathbf{CP}^2$ preserving the 2-form $\omega = \frac{dx \wedge dy}{xy}$, note that $\omega = d(\log x) \wedge d(\log y)$.

$P \in Symp$ is defined as $(x, y) \mapsto (y, \frac{1+y}{x})$.

For a form F on $\mathbf{CP}^2$ by (F) we mean it's zero locus.

T will denote the Thompson group T. It is the group of piecewise linear automorphisms of $\mathbf{Z}^2$, for precise definition and it's equivalent constructions see one of the chapters below.

$\Gamma = SL(2, \mathbf{Z})$.

H -subgroup of $Symp$ generated by Γ and P.

S is the subset of $\mathbf{Z}^2$ consisting of vectors with co-prime coordinates, or in other words $S = \mathbf{Z}^2 \backslash \bigcup_{n \geq 2} n\mathbf{Z}^2$, sometimes we call this set *primitive vectors*.

$u \wedge v$ will denote anti-symmetric bilinear product on $\mathbf{Z}^2$, normalized in a way, that $(1, 0) \wedge (0, 1) = 1$. It also gives an orientation.

For a surface A we denote $Pic(A)$ the Picard group of A which is a quotient of the group of divisors by the linear equivalence.

μ is a piecewise linear transformation of $\mathbf{Z}^2$, defined as $(x, y) \mapsto (x - min(0, y), y)$.

We use standard notation for matrices $C = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$, $I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

2. Birational automorphisms

2.1. Examples of symplectomorphisms.

Let us define an action of P, $SL(2, \mathbf{Z})$ and $\lambda \in \mathbf{C}^*$ on $\mathbf{CP}^2$ by birational automorphisms in the following way:

$$P : (x, y) \mapsto (y, \frac{1 + y}{x})$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (x, y) \mapsto (x^a y^b, x^c y^d)$$

$$\lambda : (x, y) \mapsto (\lambda x, y)$$

These transformations belong to the group $Symp$.

Using representation of $\omega = d(\log x) \wedge d(\log y)$ it is easy to verify that under these transformations ω is indeed preserved, also it is easy to see that any transformation of the sort $(x, y) \mapsto (Q(y)x, y)$ where Q is any rational function, may be obtained as a combination of these more primitive ones.

Conjecture 1.

$Symp$ is generated by transformations of these three types.

To justify this particular form of P we want to observe that $P^5 = 1$

Also we will be interested in the structure of the group H which is generated by the automorphisms of first two types. If we denote by C the matrix $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$ and by I the matrix with rows $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, than the following relations are easy to verify:

$$C^3 = I^4 = [C, I^2] = 1$$

$$PCP = I$$
$$P^5 = 1$$

One recognizes in the first row the defining relations of $\Gamma = SL(2, \mathbf{Z})$.

Conjecture 2.
This list of relations for H is complete, i.e. all the other relations are consequences of these.

2.2. Morphism $Symp \to T$.

We will provide two ways to see this morphism, which may be called tropicalization of automorphism. One is the following: take automorphism

$$\gamma : x, y \mapsto \frac{P(x,y)}{Q(x,y)}, \frac{R(x,y)}{Q(x,y)}$$

P, Q, R - are polynomials and put $x = t^a$, $y = t^b$, for $t \in (0, +\infty)$, $a, b \in \mathbf{R}$. We are interested in the behavior of this automorphism for small generic t, so we may ignore coefficients and look at exponents, and we may search the monomial in P, R with the smallest exponent and the monomial in Q with the biggest exponent. These exponents may also be found as the smallest powers of t in the development of γ in series for a small t. So we send (a, b) to the corresponding exponents and thus obtain a piecewise linear map from $\mathbf{R}^2$ to $\mathbf{R}^2$ which is actually defined on $\mathbf{Z}^2$. To illustrate this with example, let us take $P : x, y \mapsto y, \frac{1+y}{x}$.
$t^a, t^b \mapsto t^b, \frac{1+t^b}{t^a}$, the corresponding piecewise linear map is $(a, b) \mapsto (b, min(0, b) - a)$.
Another way to construct this map of groups is more geometric.

2.3. Geometry of automorphism.

Here we use standard algebraic geometry techniques to understand the geometry of a symplectic automorphism. Identifying the curves on surface with it's strict transform after blow-up, we may speak about all curves where the form ω has a pole. All these curves may be parameterized by the set S of primitive vectors of $\mathbf{Z}^2$, so the birational symplectomorphism just permutes these curves, and so defines a piecewise linear automorphism of $\mathbf{Z}^2$.
Suppose $f \in Symp$, than it is known that f may be presented in a unique way as a sequence of blow-ups and then blow-downs, namely there exists $\pi_i : X_i \to \mathbf{CP}^2$ $i = 1, 2$ and biregular isomorphism $\phi : X_1 \to X_2$ such that π_i is a composition of blow-ups and we have $f = \pi_2 \circ \phi \circ \pi_1^{-1}$.
Denote by E_1, E_2 exceptional set of π_1 and π_2 respectively, so E_i is a union of $\mathbf{P}^1$'s, $\pi_i(E_i)$ is a set of points and π_i induces bijection between $X_i \setminus E_i$ and $\mathbf{CP}^2 \setminus \pi_i(E_i)$. Remind that f is a well defined morphism except at a finite set of points(precisely at $\pi_1(E_1)$), so we may pull-back the 2-form by f everywhere except this finite set and then continue it analytically to all

surface. So $f^*\omega = \omega$ is equivalent to $\pi_1^*\omega = \phi^*\pi_2^*\omega$ on $X_1 \backslash E_1$.

The divisor corresponding to the form ω on $\mathbf{CP}^2$ is $-(X) - (Y) - (Z)$.

Let us call *chain* a union of smooth rational curves on surface: D_1, D_2, $\ldots, D_n$, such that D_i intersects only with D_{i+1} and D_{i-1} transversally at one point. $D_{n+1} = D_1$. For example the support of the divisor of ω on $\mathbf{CP}^2$ is a chain. We will consider pairs (X, D) where X - surface, D - chain on it. We start from a pair $(\mathbf{CP}^2, (X) \cup (Y) \cup (Z))$ and when $\widetilde{X}$ is a blow-up of X at point p and $\widetilde{D}$ is a strict transform of D then if p is an intersection point of two curves of a chain, we pass to $(\widetilde{X}, \widetilde{D} \cup E)$, E exceptional divisor, otherwise we pass to $(\widetilde{X}, \widetilde{D})$.

Lemma

1. π_1 is a sequence of blow-ups which are supported only on the chain
2. chains on X_1 and X_2 are isomorphic through f

Proof

First let us make local computation: fix u,v local coordinates and suppose that locally the form looks like $u^k v^l du \wedge dv$. After blow up this form will become $u^{k+l+1}(\frac{v}{u})^l du \wedge d(\frac{v}{u})$ which shows that the exceptional divisor will be a zero of order $k + l + 1$.

If we identify curves with their strict transforms, form w will be of order -1 precisely on the curves of chains, so the equality $\pi_1^* w = \pi_2^* w$ implies the second statement of the lemma.

To prove the first statement, remark that if we blow-up something else than a point on the chain, then the order of w at the exceptional divisor becomes positive, so as the positive parts of divisors of $\pi_1^* w$ and $\pi_2^* w$ are isomorphic we may contract them simultaneously.

Corollary. π_i involves blow-ups at two types of points: first at the points of intersection of curves of the chain, thus we enlarge chains;

second - at the interior points of curves of the chain.

Denote by S the set of curves of the chain identified with their strict transforms. Now we realize S as subset of $\mathbf{Z}^2$. Send curves $(X = 0)$, $(Y = 0)$, $Z = 0$ on $\mathbf{CP}^2$ to points $(1, 0)$, $(0, 1)$, $(-1, -1)$ respectively. and use the following rule: if the curves C_i, C_{i+1} on some chain go to $u, v \in \mathbf{Z}^2$, then send the exceptional curve appearing after the blow-up of their intersection point to $(u + v)$. This procedure gives a bijection between curves on chains and the primitive vectors in $\mathbf{Z}^2$.

Birational symplectic automorphism takes one chain to another, so it takes corresponding elements in S, say $s_1, \ldots, s_n$, to some other elements, preserving cyclic order. But if we blow-up the intersection point of curves, corresponding to s_i, s_{i+1} then the exceptional curve(which is corresponding to $s_i + s_{i+1}$) will go to the exceptional curve of the blow-up of intersection

of images of s_i and s_{i+1}. This argument shows that the action on S is linear between s_i and s_{i+1}, so we have constructed a piecewise linear automorphism of S and thus of $\mathbf{Z}^2$, which is an element of the Thompson group T.
In the appendix there is a presentation of T in terms of images of C and P. We can check, that $C, I \in SL(2, \mathbf{Z})$ go to corresponding elements in T. Element $(x, y) \mapsto (y, \frac{1+y}{x})$ maps to L.

3. THOMPSON GROUP T

Here we gather some information about group T. The classical reference is [**?**].

This group may be interpreted in different ways, as:
- piecewise linear automorphisms of $\mathbf{Z}^2$
- piecewise linear dyadic homeomorphisms of circle
- pairs of binary trees with cyclic bijection between their leaves
- pairs of regular trivalent trees with fixed oriented edge and cyclic bijection between their leaves
- piecewise projective automorphisms of circle
Here we give five definitions of T, sketching the way to pass from one definition to another.

Let us say that the set $\{s_1, \ldots, s_n\} \subset S$ is cyclically ordered if an hour hand attached to $(0, 0)$ will meet them in this order: $s_1, s_2, s_3, \ldots, s_n, s_1, s_2 \ldots$. They give a decomposition of $\mathbf{Z}^2$ into cones $\mathbf{Z}_{\geq 0}s_i + \mathbf{Z}_{\geq 0}s_{i+1}$. We require that $s_{i+1} \wedge s_i = 1$ and consider the bijections of S preserving cyclic order f, such that $as_i + bs_{i+1}$ is mapped to $af(s_i) + bf(s_{i+1})$, for a, b - non-negative. We call such bijection a piecewise linear automorphism of $\mathbf{Z}^2$.

Definition1 The Thompson group T is a group of piecewise linear automorphisms of $\mathbf{Z}^2$.

Dyadic number is a number of the form $\frac{p}{2^q}$, $p, q \in \mathbf{Z}$. Circle is an interval $[0, 1]$ with $0 = 1$. Dyadic automorphism of circle would be a piecewise-linear automorphism, that sends dyadic numbers to dyadic and whose derivatives on the intervals of linearity are 2^k, where k - integer numbers.

Definition2 The Thompson group T is a group of dyadic automorphisms of a circle.

Actually if we put $S^1 \ni 0 = 1 \mapsto (1, 0) \in S$, $\frac{1}{2} \mapsto (0, 1)$, $\frac{3}{4} \mapsto (-1, -1)$, then we can extend this into bijection between dyadic points and S preserving cyclic ordering. Then it is possible to check that the automorphisms of

the set of dyadic points, coming from dyadic automorphisms, and automorphisms of S, coming from PL automorphisms of $\mathbf{Z}^2$, coincide.

Binary tree has a root, and every vertex has either no or two descendants - left and right. Vertices with no descendants are called leaves, they are naturally cyclically ordered. Such a tree gives a decomposition of the interval: we attach $[0,1]$ to the root. If vertex has an interval $[p,q]$ attached, we associate $[p, \frac{p+q}{2}]$ to it's left descendant and $[\frac{p+q}{2}, q]$ to it's right descendant. Clearly enough all such intervals are of length $\frac{1}{2^k}$ and intervals attached to leaves give a decomposition of $[0,1]$.

Definition3 The Thompson group T has elements represented by equivalence classes of pairs of binary trees (R,U) with the equal number of leaves, and a bijection between the leaves of R and the leaves of U, preserving the cyclic order. Attaching simultaneously two descendants to the leaf on R and to the corresponding by the bijection leaf of U gives an equivalent pair of trees, so it would give the same element of T. A composition of elements (R,S) and (M,N) is the following: find a binary tree Z, which contains all the vertices of S and M, then find representatives of the equivalence classes of (R,S) and (M,N), which look like (X,Z) and (Z,Y), by adding descendants to appropriate leaves, completing S and M to Z. The composition of (X,Z) and (Z,Y) is (X,Y), with the bijection on leaves induced.
If we attach intervals to a tree (R,U) as explained before, then mapping intervals of leaves of R to intervals of corresponding leaves of U gives a dyadic automorphism of a circle.

Looking at the binary tree as at the graph, let us remove a root and it's two adjacent edges and join root's left and right descendant by an oriented edge going from left to right. All other edges stay unoriented.
Let Υ be a set of connected planar trees with vertices of valence 3 or 1 and one oriented edge (vertices of valence one are called leaves and they are cyclically ordered). Consider the set Θ of pairs (A,B), $A,B \in \Upsilon$ with a bijective identification of leaves, preserving the cyclic order. Put an equivalence relation on Θ generated by the following: if we add two leaves to the leaf of A and two leaves to the corresponding leaf of B, the new pair of trees with naturally induced bijection of leaves is equivalent to (A,B).

Definition4 The set of equivalence classes of Θ under this equivalence relation and composition given on representatives $(A,B) \cdot (B,C) \mapsto (A,C)$ has a well-defined structure of group, and is called the Thompson group T.

Given an element $A \in \Upsilon$, we will attach to it a union of triangles in the unit disc $D = \{|z| \leq 1\}$. So to every vertex of A we attach a triangle, and for any two vertices joined by an edge the corresponding triangles have

a common edge. The edges of all such triangles will be geodesic and the vertices will be rational in the upper half-plane model of a disc.

To the vertex from which the oriented edge starts we associate the triangle with vertices $-i, i, -1$, to the vertex where the oriented edge goes we associate the triangle with vertices $-i, 1, i$. Next, suppose that we associate the triangle abc with a vertex, and we want to construct a triangle adjacent to the edge bc. We just need to choose a unique hyperbolic automorphism of the disc, that sends $-1, -i, i$ to a, b, c in this order, then c, b and the image of 1 will give the vertices of the new triangle. Triangles corresponding to leaves will have internal edges - those that correspond to the only edge of the leaf and by which this triangle is attached to previously constructed ones. Internal edges form a polygon, inscribed in the disc, so if we remove the polygon, the rest is the union of half discs, each arc having a point. So a pair of trees, corresponding to an element of Θ, gives a pair of sequences of half discs, thus a piecewise projective automorphism of a circle.

Definition5 Thompson group T is a group of piecewise projective automorphisms of a circle.

Although we consider just finite binary trees, infinite full binary tree will correspond to the Farey trianguation of a disc.

4. Presentation in the Picard group

The aim of this section is to construct a linear representation of H in an infinite dimensional vector space. First we construct a pro-scheme X, which is a projective limit of surfaces, blow-ups of $\mathbf{CP}^2$ in appropriate points. Group H generated by Γ and P will act on it by automorphisms. We make sence of the Picard group of X, and H will act on this free abelian group, preserving a bilinear symmetric product of signature $(1, \infty)$.

$\Gamma = SL(2, \mathbf{Z})$ is a subgroup of H, so first we apply the same procedure to it and obtain object Y, which may be called universal toric variety.

At the end we just give simple description of action of Γ and $\mu = I^{-1}P^{-1}$ on space V, which gives a representation of H.

4.1. **Universal toric variety.** In this section we will develop the necessary details about the projective limit of all toric surfaces. Actually, denote by Y the pre-scheme, which is a projective limit of blow-ups of $\mathbf{CP}^2$ at the points of intersection of chains. The group Γ will act on it as the group of automorphisms. Then we compute the Picard group and find the cones of ample and effective divisors.

Actually if $f : A \to B$ is the blow-up of a smooth surface B at a point then we have the pullback $f^* : Pic(B) \to Pic(A)$ and if (E) is the class of the exceptional divisor we have $Pic(A) = f^*Pic(B) \oplus \mathbf{Z}(E)$ and we have

also the projection morphism $f_* : Pic(A) \to Pic(B)$ which is just a forgetting of the exceptional set. So we are allowed to consider an injective and a projective limit of Pic groups, note them Pic and $\widehat{Pic}$ respectively, moreover the former is embedded in later.

Recall that S is the set of points in $\mathbf{Z}^2$ with co-prime coordinates. Actually S parameterizes the rational curves coming from chains. Let $PL(S)$ be the space of piecewise linear function from S to $\mathbf{Z}$, and let $Fun(S)$ be the space of all functions from S to $\mathbf{Z}$. Inside PL and Fun we have two-dimensional space Lin of linear functions.

Lemma $Pic(Y) = PL/Lin$, $\widehat{Pic} = Fun/Lin$

To explain, why this Lin appears, note that in $Pic(CP^2)$ three divisors $(X = 0)$, $(Y = 0)$, $(Z = 0)$ are equal. The corresponding elements in $Pic(Y)$ are functions, generated by there values in the points $(1, 0)$, $(0, 1)$, $(-1, -1)$ and extended by linearity to the rest of the S. So $(X = 0)$ has corresponding values 1,0,0. $(Y = 0)$ has 0,1,0, and $(Z = 0)$ has 0,0,1. So in $Pic(Y)$ these three functions are equal, which is equivalent for Lin to be trivial.

The action $SL(2, \mathbf{Z})$ on $Pic(Y)$ and on $\widehat{Pic}(Y)$ is induced from natural action on PL and Fun, space of linear functions and Lin is preserved.

There is a non-degenerate pairing $Pic(Y) \times \widehat{Pic}(Y) \to \mathbf{Z}$ which would be defined later. Also there is an embedding $Pic(Y) \subset \widehat{Pic}(Y)$

The important part of the structure is the cone of effective divisors $Eff_Y \subset \widehat{Pic}(Y)$ generated by positive functions on S. The dual to it is the cone of ample divisors $Amp_Y \subset Pic(Y)$ which consists of convex functions. Both cones are preserved under the action of the group Γ.

Let F be a piecewise linear function from S to $\mathbf{Z}$. For $a \in S$ we will introduce an index $d(F, a)$ which will measure how much F is far from being linear. So in the neighborhood of a choose u, v such that $u \wedge a = a \wedge v = 1$(so a is between u and v). We ask that F is not linear at most at a on this interval, which can be realized by adding a to u and v sufficiently many times. Then obviously $u + v = ka$ for $k \in \mathbf{Z}$. Put $d(F, a) := F(u) + F(v) - kF(a)$ and call this value an index of the function F at a. Easy to check that it does not depend on the choice of u, v, and it equals to 0 at all points where F is linear. d is linear in the first argument.

Let us describe a pairing between $Pic(Y)$ and $\widehat{Pic}(Y)$. Suppose F is a piecewise linear and G is any function on S. The pairing is defined as follows:

$$< F.G >= \sum_{\alpha \in S} d(F, \alpha) G(\alpha)$$

The reason to introduce Y is that now Γ acts by automorphisms on it. Next we find a pre-scheme where H acts by automorphisms.

4.2. X**.** Notice that when we take a finite sequence of blow-ups of $\mathbf{CP^2}$ at chain points, we come out with a canonical coordinate on each exceptional divisor. First X/Y, Y/Z, Z/X are coordinates on divisors $Z = 0$, $X = 0$, $Y = 0$. Then if we blow-up a point of intersection of two rational curves with canonical coordinates say a, b on curves α, β at a point $a = 0, b = \infty$, choose local coordinates A, B at this point, such that $A\mid_\alpha = a$, $A\mid_\beta = 0$, $B\mid_\alpha = 0$, $B\mid_\beta = 1/b$. Then we take A/B as canonical coordinate on the exceptional divisor.

Proposition H preserves canonical coordinates on the exceptional divisors coming from chains.

Proof It is sufficient to prove this statement for P, C the generators of H, which may be done by direct computation.

The corollary is that every rational chain curve in Y has canonical coordinate on it, so we may consider the set of points Ω with coordinate -1, and this set is preserved by H. The automorphism $(x, y) \mapsto (y, \frac{1+y}{x})$ involved the blow-ups of points in this set. So let us consider projective limit X of surfaces, obtained by blow-ups at Ω. After a blow-up at a point $p \in \Omega$ replace p by intersection point of divisor of chain on which p was supported and the exceptional divisor. Then one may blow-up again in new p and so on. If $s \in S$ corresponds to the divisor supporting p, denote δ_s the exceptional divisor. Denote δ_s^2 the exceptional divisor of the blow-up at the intersection point of s and δ_s etc. Denote by X the projective limit of blow-ups of this kind and actually this is the object where H acts by automorphisms.

Let $\sum_{s,k} \mathbf{Z}\delta_s^k$ be a free abelian group generated by δ_s^k and $\prod_{s,k} \mathbf{Z}\delta_s^k$ be a free product over indices $s \in S$ and $k \in \mathbf{Z_{>0}}$.

Proposition $Pic(X) = Pic(Y) \oplus \sum_{s,k} \mathbf{Z}\delta_s^k$.
$\widehat{Pic}(X) = \widehat{Pic}(Y) \oplus \prod_{s,k} \mathbf{Z}\delta_s^k$.

The pairing between $Pic(X)$ and $\widehat{Pic}(X)$ is extended from that on Y by saying that δ_s^k are orthogonal to each other and to $Pic(X)$ and $\delta_s^k . \delta_s^k = -1$.

Now group H generated by L and C is acting by orthogonal transformations on $Pic(X)$ (L denotes the inverse of P). We just find it more convenient to describe the action of L on $Pic(X)$.

Remind that L is given by $(x, y) \mapsto (\frac{1+x}{y}, x)$ which is the inverse of $P : (x, y) \mapsto (y, \frac{1+y}{x})$ considered before.

Denote by A the piecewise linear function $A(x, y) = max(0, -y)$. Actually it is not linear only at $(1, 0)$ and $(-1, 0)$.

Lemma
L acts as follows:

$\delta_s^k \mapsto \delta_{L(s)}^k$, for $k \geq 1$, $s \neq (0, \pm 1)$

$\delta_{(0,-1)}^k \mapsto \delta_{(1,0)}^{k+1}$

$\delta_{(0,1)}^k \mapsto \delta_{(-1,0)}^{k-1}$ for $k \geq 2$

$\delta_{(0,1)}^1 \mapsto -\delta_{(1,0)}^1 + A$

for $F \in PL(S)$ we have $F \mapsto F \circ L^{-1} - F((0,-1))\delta_{(1,0)}^1 + F((0,1))(-\delta_{(1,0)}^1 + A)$

4.3. **The presentation of H.** In $\widehat{Pic}(Y) \subset \widehat{Pic}(X)$ we may consider functions which take value 1 at some point and 0 at all the others. Geometrically they will define some curve of the chain. As the group H preserve curves of the chain, the subspace of associated functions will also be preserved by it. So H will act on the space V dual to it. To describe it, let us do the following. For any piecewise linear function $F \in PL$ we associate $F' := F - \sum_i d(F, s_i)\delta_{s_i}^1$.

Lemma The space V dual to the space of chain curves is generated by F' and $\delta_\alpha^k - \delta_\alpha^{k+1}$

So $V \subset Pic(X)$ is invariant under group H, which may be also checked directly. the action looks quite simple in the appropriate basis. First let $F' = F - \sum_\alpha d(\alpha, F)\delta_\alpha^1$ for any piecewise linear function $F \in PL$. Denote also $e_\alpha^k = \delta_\alpha^k - \delta_\alpha^{k+1}$. Then such functions F' modulo linear functions are encoded by their indexes of non-linearity. So introduce the free abelian groups $B = \bigoplus_{\alpha \in S} b_\alpha$ and $E = \bigoplus_{k=1, \alpha \in S}^\infty e_\alpha^k$. Now to any F' we associate $\sum d(\alpha, F)b_\alpha \in B$. Such elements generate subgroup $B_0 \subset B$ of corank 2. Actually B_0 consists of $\sum n_\alpha b_\alpha$ which are in the kernel of the map $b \to \mathbf{Z^2}$ defined by $b_\alpha \mapsto \alpha$. This is the consequence of the fact, that pairings of function F with linear functions are 0.

So H acts on the $V = B_0 \oplus E$.

The action of Γ is natural on the indices of e and b. The action of the element $\mu = I^{-1}L$ may be called the mutation applied to the vector $v = (0, 1)$. By analogy we would say, that $\gamma \circ \mu \circ \gamma^{-1}$ is the mutation applied to the vector v if $v = \gamma((0, 1))$. Here is how the mutation μ applied to the vector $v \in S$ acts:

$$b_v \mapsto -b_{-v}$$

$$b_{-v} \mapsto e_{-v} + b_{-v}$$

$$b_w \mapsto b_{\mu_v(w)} \text{ if } w \wedge v > 0$$

$$b_w \mapsto b_{\mu_v(w)} + (v \wedge w)b_{-v} \text{ if } w \wedge v < 0$$

$$e_v \mapsto b_v + b_{-v}$$

$$e_v^k \mapsto e_v^{k-1}$$

$$e_{-v}^k \mapsto e_{-v}^{k+1}$$

$$e_w \mapsto e_{\mu_v(w)}$$

Let us make a change of basis: if $v \in S$ and k are positive integer then let

$$p_{kv} := e_{(k-1)v} + 2e_{(k-2)v} + ... + (k-1)e_v + kb_v$$

Now V can be described as the kernel of the map $\bigoplus_{v \in \mathbf{Z^2} \setminus \{(\mathbf{0,0})\}} \mathbf{Z}p_v \to \mathbf{Z^2}$, where p_v maps to v.

The action of μ_v is the following: first denote $A_v(w) := (w \wedge v)$ if $(w \wedge v) > 0$, 0 if $(w \wedge v) < 0$ and -1 if it's $= 0$

Then $\mu_v(p_w) := p_{\mu_v(w)} + A_v(w)p_{-v}$.

Note that $\mu_v(kv) := (k-1)v$ and $p_0 = 0$, so $\mu_v(p_{kv}) := p_{(k-1)v} - p_{-v}$.

Here we summarize the previous calculations in more compact formulas to obtain a presentation of H in V.

Let $\Lambda = \mathbf{Z^2} \setminus (0,0)$ be a set , and $W = \bigoplus_{i \in I} \mathbf{Z}e_i$ be a $\mathbf{Z}$-module. Remind the action of μ on Λ is given by the following: $\mu : i = (x,y) \mapsto (x - min(0,y), y)$.

Now we define an action of μ on $W[q]$ - free abelian group of polynomials in q with coefficients in W.

$$e_{(x,y)} \mapsto e_{(x,y)} + (1-q)ye_{(-1,0)} \text{ if } y > 0$$

$$e_{(x,y)} \mapsto e_{(x-y,y)} + qye_{(-1,0)} \text{ if } y < 0$$

$$e_{(x,0)} \mapsto e_{(x-1,0)} - e_{(-1,0)}$$

We have a map $W \to \mathbf{Z^2}$, $e_{(x,y)} \mapsto (x,y)$, denote by V it's kernel.

Lemma

V is an invariant space for the action of μ and Γ and is a presentation of H

Consider the pull-back of the wedge product from $\mathbf{Z^2}$ to W, and denote it by $\wedge$ too.

Lemma

$\wedge$ is preserved by the action of μ and Γ.

4.4. **Cluster mutations.** By calling a piecewise linear transformation μ mutation, we made an allusion to a cluster mutation, so let us try to justify this similarity.

To a given cluster $\underline{t}$ we associate a vector space $V(\underline{t}) = \bigoplus_{\lambda \in \Lambda(\underline{t})} \mathbf{C}e_\lambda$, where $\Lambda(\underline{t})$ is the set of indices for cluster variables at this cluster, we suppose $b_{ij}(\underline{t})$ is an antisymmetric matrix of exponents. Suppose a mutation μ_i at variable $i \in \Lambda(\underline{t})$ relates a cluster $\underline{t}$ to $\underline{t}'$, then we have a bijection $\sigma_i : \Lambda(\underline{t}) \to \Lambda(\underline{t}')$. Let us define a map $\mu_i : V(\underline{t}') \to V(\underline{t})$ in the following way:

$$e_{\sigma(k)}(\underline{t}') \mapsto e_k + max(b_{ik}(\underline{t}), 0)e_i$$

$$e_{\sigma(i)(\underline{t}')} \mapsto -e_i$$

It is easy to check that $b(\underline{t})$ which may be considered as an element of $\bigwedge^2 V^*(\underline{t})$ taking value b_{ij} at $e_i \wedge e_j$ transforms in a way, consistent with the

transformation of mutation matrix.

Let us now set $\Lambda = \mathbf{Z}^2$ and let us allow mutations only at the primitive vectors $S \subset \Lambda$. Take all the seeds to be isomorphic and set $b_{ij} = i \wedge j$, so rather than having different seeds we would act by mutations on a seed data. First of all σ_v for $v \in S$ acts on Λ as follows: $\sigma_v : w \mapsto w - min(v \wedge w, 0)v$ if $w \wedge v \neq 0$

$\sigma_v : w \mapsto w - v$ if $w = kv$

It is an automorphism of Λ. It's easy to see that $\sigma_{\gamma(v)} = \gamma \circ \sigma_v \circ \gamma^{-1}$ for any $\gamma \in \Gamma$. We may also denote $\mu := \sigma_{(1,0)}$.

Recall the alternative description of the Thompson group T given in the Appendix: $T = < L, C'|I = LC'L, C'^3 = I'^4 = L^5 = (C'I'L)^7 = 1, I'^2C' = C'I'^2 >$. We put dashes to distinguish these elements from generators of $SL(2, \mathbf{Z})$.

Let us invert this presentation, i.e. put $P = L^{-1}$, $C = C'^{-1}$, $I = I'^{-1}$. Then we have:

$$T = < P, C|I = PCP, C^3 = I^4 = P^5 = (PIC)^7 = 1, I^2C = CI^2 >$$

Now we can send the group $\Gamma = SL(2, \mathbf{Z})$ to T by mapping $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ to C, and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ to I. Denote $U = CI = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\mu = IP$. In this notations we deduce the following presentation of T:

Theorem T is generated by Γ and μ, subject to following relations:
$(I^2\mu)^2 = U^{-1}$
$(I^{-1}\mu)^5 = 1$
$(I\mu)^7 = 1$
all the other relations are the consequences of these.

Now the vector space V_0 associated to the cluster as explained above is $V_0 = \bigoplus_{s \in \Lambda} \mathbf{C}e_s$, and the action of μ_v is:

$$e_w \mapsto e_{\sigma_v(w)} + max(v \wedge w, 0)e_{-v}$$

$$e_v \mapsto -e_{-v}$$

And this is the formula for the action of μ in the representation of H, except for e_w when w is collinear with v.

5. Quantization

Suppose now that variables x, y are not commuting. Introduce formal parameter q commuting with both x and y and let $xy = qyx$.

Consider the following transformations:

$$P : x, y \mapsto y, qx^{-1}(1 + y)$$

$$C : x, y \mapsto x^{-1}yq, x^{-1}q$$

$$I : x, y \mapsto y^{-1}q, x$$

Then, supposing the conjecture 2, it is easy to verify that all known relations in group H hold true.

One way to interpret this is the following abstract idea, inspired by mirror symmetry: we have the family of non-commutative P^2's, parameterized by q, which for $q = 1$ degenerate to usual $\mathbf{CP^2}$ with Poisson bracket, indicating the direction of noncommutative P^2's. The general framework $[KS]$ predicts that there is a piecewise-linear object which may be associated to a degeneration, such that the automorphisms of the family act as piecewise linear automorphisms of this limit object. In our case, apparently the limit object is a circle with the set of points S and a piecewise linear structure coming from $\mathbf{Z^2}$. So the noncommutative automorphisms P, C, I act on this S^1 in a piecewise-linear way.

6. Appendix. Alternative description of the Thompson group T

First recall the standard description of T group, given by generators:

$$A, B, C$$

notations: $R = A^{-1}CB$, $X_2 = A^{-1}BA$, $P = A^{-1}RB$
and relations

$$R = B^{-1}C \ (1)$$
$$RX_2 = BP \ (2)$$
$$CA = R^2 \ (3)$$
$$C^3 = 1 \ (4)$$
$$BA^{-1} \text{ commutes with } X_2 \ (5)$$
$$BA^{-1} \text{ commutes with } A^{-1}X_2A \ (6)$$

as consequences we have the following relations:
$R^4 = 1$, $P^5 = 1$, $(P^2 X_2^{-1})^3 = 1$, $(PX_2)^4 = 1$, $(X_2^2 P^{-2})^4 = 1$.
In what follows we will use the following procedure: suppose we are given two groups G_1, G_2, which are isomorphic and defined by generators and relations, with N_1, N_2 - normal subgroups, such that isomorphism takes N_1 to N_2. Then we will add some element to N_1 and the conjugate of it's image by isomorphism to N_2. We call elements of N_i relations, so that at the end we obtain isomorphic groups G_1/N_1 and G_2/N_2 which give two presentations of the same group in terms of generators and relations. If one of the relations has the generator from one side, and the expression, doesn't depending on this generator from the other side, we will call it a notation, and will allow ourselves to remove this generator and this relation and replace in all the other relations this generator by this expression. We add the relations to N_1 in order to obtain as G_1/N_1 the known representation of T and G_2/N_2 will give another representation of it.

Now we'll search for the representation of T-group in terms of R and C.

So start from groups $G_1 =< A, B, C | R = B^{-1}C, CA = R^2 >$ and $G_2 =<$ $R, C >$ which are isomorphic, normal groups N_i take trivial. Thus $A = C^{-1}R^2$, $B = CR^{-1}$. Add $C^3 = 1$ to both N_1 and N_2. Adding $R^{-1}A^{-1}CB$ to N_1 is equivalent to adding R^4 to N_2. $X_2 = A^{-1}BA = R^{-2}CCR^{-1}C^{-1}R^2$, $P = A^{-1}RB = R^{-2}CRCR^{-1}$, so adding $RX_2(BP)^{-1}$ to N_1 using $C^3 = R^4 = 1$ becomes equivalent to adding $(CR)^5$ to N_2. So the group generated by A, B, C with relations $(1) - (4)$ is isomorphic to the $< R, C | C^3 = R^4 = (CR)^5 = 1 >$.

Now introduce $\alpha = BA^{-1} = CR^{-1}R^{-2}C = CRC$. Again $X_2 = A^{-1}BA = R^2(CRC)^{-1}R^2 = (R^2\alpha R^2)^{-1}$, $A^{-1}X_2A = R^2CR^2\alpha^{-1}R^2C^{-1}R^2$, so we obtain another description of T-group:

generators

$$R, C$$

Notations: $\alpha = CRC$, $\beta = R^2C^{-1}R^2$

Then relations are:

$$R^4 = 1$$

$$(RC)^5 = 1$$

$$C^3 = 1$$

$$\alpha \text{ commutes with } (R^2\alpha R^2)$$

$$\alpha \text{ commutes with } (\beta^{-1}\alpha\beta)$$

Consider now the element $L = APA^{-1} \in T$.

Theorem

$$T =< L, C | I = LCL, C^3 = I^4 = L^5 = (CIL)^7 = 1, I^2C = CI^2 >$$

Proof

Again as in previous considerations let us start from isomorphic groups $G_1 =< R, C | C^3 = (RC)^5 = 1 >$ and $G_2 =< L, C | C^3 = L^5 = 1 >$. The isomorphism is established by saying that $L \mapsto (RC)^2$ and $R \mapsto L^{-2}C^{-1}$

Make a notation $I = LCL$ in G_2. $R = FC^{-1} = L^{-2}C^{-1} = (CL^2)^{-1}$ so $R^4 = 1$ becomes $(CL^2)^{-4} = 1$ or by conjugating $1 = L(CL^2)^4L^{-1} = (LCL)^4 = I^4 = 1$. So far we have established the following group isomorphism:

$$< R, C | C^3 = R^4 = (RC)^5 = 1 >=< L, C | C^3 = L^5 = (LCL)^4 = 1 >$$

$L \mapsto (RC)^2$, $C \mapsto C$, $R \mapsto L^{-2}C^{-1}$

Now consider first commutator(we use $[X, Y] = X^{-1}Y^{-1}XY$, $X^Y = Y^{-1}XY$, also by definition of $I = LCL$ we have $LCI = ICL$):

$R^2\alpha R^2 = L^{-2}C^{-1}L^{-2}C^{-1}CL^{-2}L^{-2}C^{-1}L^{-2}C^{-1} == L^{-2}C^{-1}L^{-1}C^{-1}L^{-2}C^{-1} = L^{-1}I^{-1}C^{-1}L(CL^{-2})^{-1}$, so $[\alpha, R^2\alpha R^2] \mapsto [CL^{-2}, L^{-1}I^{-1}C^{-1}L(CL^{-2})^{-1}] = [L^{-1}I^{-1}C^{-1}L, (CL^{-2})^{-1}] = [I^{-1}C^{-1}, (IL)^{-1}]^L = (CIILI^{-1}C^{-1}L^{-1}I^{-1})^L = (CI^2L(LCI)^{-1}I^{-1})^L = (CI^2L(ICL)^{-1}I^{-1})^L = [C^{-1}, I^{-2}]^L$, so the first commutator relation in G_1 is equivalent to $[C, I^2] = 1$ in G_2.

Note now, that the elements C, I have the relations $C^3 = I^4 = 1$, $[C, I^2] = 1$, actually they generate the $SL(2, \mathbf{Z})$ inside T, where C corresponds to the

matrix with rows $(-1, 1), (-1, 0)$, and I - to $(0, -1), (1, 0)$.

Let us deal now with the second commutator, we are interested in it up to conjugation, so we may now conjugate it's arguments. $R^2 = R^{-2} = CL^2CL^2$, $\beta = R^2C^{-1}R^2 = CL^2CL^{-1}CL^2$, $\beta^{-1}\alpha\beta = L^{-2}C^{-1}LC^{-1}L^{-2}C^{-1}CL^{-2}CL^2CL^{-1}CL^2 = L^{-1}(LCL)^{-1}L^2C^{-1}LCL^2CLL^{-3}LCLL = L^{-1}I^{-1}L^2C^{-1}I^2L^2IL = (L^2C^{-1}I^2L^2)^{IL}$ this commutes with $\alpha = CL^{-2}$. Let us write $V = C^{-1}I^2 = I^2C^{-1}$, which will represent the matrix $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$.

$\alpha^{(IL)^{-1}} = ILCL^{-2}L^{-1}I^{-1} = I^2LI^{-1} = I^2C^{-1}L^{-1} = VL^{-1}$. So the second commutator after conjugating it by $(IL)^{-1}$ becomes $[\beta^{-1}\alpha\beta, \alpha]^{(IL)^{-1}} = [(\beta^{-1}\alpha\beta)^{(IL)^{-1}}, \alpha^{(IL)^{-1}}] = [L^2C^{-1}I^2L^2, VL^{-1}] = L^{-2}V^{-1}L^{-2}LV^{-1}L^2VL^2VL^{-1}$. Notice that $(CIL)^2 = CI(LCI)L = CI(ICL)L = VL^2$, so commutating now by L^{-2} we get $V^{-1}L^{-1}V^{-1}L^2VL^2VL^2 = V^{-1}L^{-1}V^{-2}(VL^2)^3 = CI^2(CL)^{-1}(CIL)^6 = (CIL)^7$.

So the proof of the theorem is accomplished.

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