CREMONA TRANSFORMATIONS ASSOCIATED WITH THE CHORDS OF A TWISTED CUBIC

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1. **Introduction.** A Cremona space transformation is a one–one relation between generic points P and P' of two spaces S and S', respectively [1; 155]. Such transformations are birational. The spaces S and S' are supposed to exist independently unless the contrary is specified. However, S and S' may be superposed. This gives rise to additional properties of the transformation which are of interest, for example, invariant and involutory elements, self-corresponding elements; in particular, it brings into being the associated complex of lines joining homologous points. Under certain circumstances the complex may reduce to a congruence. In the case of involutorial transformations it is known [1; 181] that if the complex reduces to a congruence this congruence is of the first degree and consists of either:

(i) the lines through a point,

(ii) the lines meeting a line l and a curve of degree m which meets l in $m - 1$ points, or

(iii) the chords of a cubic curve.

Involutions having associated congruences of types (i) and (ii) have been discussed by the author in two previous papers [2], [3]. The present paper is concerned with Cremona transformations, both involutorial and non-involutorial, having associated congruences which are the chords of a twisted cubic. As in the two papers just mentioned the discussion is almost entirely analytic.

2. **Definition of the involution.** Consider a twisted cubic r and a pencil of surfaces

$$| F_{2n+2} |: r^n g_{n^2+8n+4} ,$$

of order $2n + 2$, in which the cubic r is contained n times. Through a generic point $P(y)$ there passes a single F of $| F |$, and also through P there is a unique line t belonging to the congruence of chords of r. The line t meets F a second time in a point $Q(x)$, the image of $P(y)$ under the transformation so defined. The residual base curve of $| F |$ has been denoted by g, is of order $n^2 + 8n + 4$, and is considered to be non-composite. It will be shown that r and g are fundamental curves of the involution which is of order $4n + 9$.

3. **Equations of the involution.** Let us take the equation of the twisted cubic r as

$$(1) \qquad x_1 : x_2 : x_3 : x_4 = h^3 : h^2 : h : 1$$

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and the pencil of surfaces $| F |$ as

$$| F | \equiv U - uU' = 0, \tag{2}$$

where

$$\begin{aligned} U_{2n+2} &= (ax)H_1^n + (bx)H_2^n + (cx)H_3^n \\ U'_{2n+2} &= (a'x)H_1^n + (b'x)H_2^n + (c'x)H_3^n \\ (ax) &= \sum_{i,j=1}^{4} a_{ij}x_ix_j , \qquad a_{ij} = a_{ji} \\ (a'x) &= \sum_{i,j=1}^{4} a'_{ij}x_ix_j , \qquad a'_{ij} = a'_{ji} \\ H_1 &= x_1x_3 - x_2^2 , \qquad H_2 = x_1x_4 - x_2x_3 , \qquad H_3 = x_2x_4 - x_3^2 , \end{aligned} \tag{3}$$

and so on.

Through a generic point $P(y)$ there passes one F of $| F |$ having parameter $u = U(y)/U'(y)$ and equation

$$U(x)U'(y) - U'(x)U(y) = 0. \tag{4}$$

Also through $P(y)$ there passes a unique line t belonging to the congruence and meeting r in the two points

$$(h_1^3 , h_1^2 , h_1 , 1) \quad \text{and} \quad (h_2^3 , h_2^2 , h_2 , 1) \tag{5}$$

in which h_1 and h_2 are the roots of the quadratic [4; 11]

$$H_3h^2 - H_2h + H_1 = 0. \tag{6}$$

The line through the points (5) meets the plane $x_3 = 0$ in the point whose coordinates are

$$(H_1H_2 , H_1H_3 , 0, - H_3^2). \tag{7}$$

Since the points (5) have coordinates which are irrational expressions in y we determine t as the line through $P(y)$ and the point (7). This line meets the surface (4) in one residual point $Q(x)$ having coordinates

$$\begin{aligned} \rho x_1 &= \bar{L}y_1 + \bar{K}H_1H_2 \\ \rho x_2 &= \bar{L}y_2 + \bar{K}H_1H_3 \\ \rho x_3 &= \bar{L}y_3 \\ \rho x_4 &= \bar{L}y_4 - \bar{K}H_3^2 , \end{aligned} \tag{8}$$

where

$$\begin{aligned}
\bar{L}_{4n+10} &= UW' - U'W, \qquad \bar{K}_{4n+7} = U'V - UV' \\
V_{2n+5} &= \{aZy\}H_1^n + \{bZy\}H_2^n + \{cZy\}H_3^n \\
V'_{2n+5} &= \{a'Zy\}H_1^n + \{b'Zy\}H_2^n + \{c'Zy\}H_3^n \\
W_{2n+8} &= \{aZH\}H_1^n + \{bZH\}H_2^n + \{cZH\}H_3^n \\
W'_{2n+8} &= \{a'ZH\}H_1^n + \{b'ZH\}H_2^n + \{c'ZH\}H_3^n \\
\{aZy\} &= (aZ_1)y_1 + (aZ_2)y_2 + (aZ_3)y_3 + (aZ_4)y_4 \qquad (9)\\
\{aZH\} &= (aZ_1)H_1H_2 + (aZ_2)H_1H_3 - (aZ_4)H_3^2 \\
(aZ_1) &= a_{11}H_1H_2 + a_{12}H_1H_3 - a_{14}H_3^2 \\
(aZ_2) &= a_{21}H_1H_2 + a_{22}H_1H_3 - a_{24}H_3^2 \\
(aZ_3) &= a_{31}H_1H_2 + a_{32}H_1H_3 - a_{34}H_3^2 \\
(aZ_4) &= a_{41}H_1H_2 + a_{42}H_1H_3 - a_{44}H_3^2 ,
\end{aligned}$$

and so on.

However, both $\bar{L}$ and $\bar{K}$ contain the factor y_3. Writing $\bar{K} = y_3K$ and $\bar{L} = y_3L$ and removing the factor y_3 from the right-hand side of equations (8) they become

$$(10) \qquad \begin{aligned}
\rho x_1 &= Ly_1 + KH_1H_2 \\
\rho x_2 &= Ly_2 + KH_1H_3 \\
\rho x_3 &= Ly_3 \\
\rho x_4 &= Ly_4 - KH_3^2 .
\end{aligned}$$

As a consequence of the fact that the point (7) is an ordinary point of the involution it is found that y_3 will factor out of the right-hand members of equations (10). It should be noted that y_3 is not a factor of L or of K individually. When this factor y_3 is also discarded the equations (10) become the equations of the involution under consideration which is now seen to be of order $4n + 9$.

4. **Images of fundamental curves.** The cubic r is the intersection of the quadrics H_1, H_2, H_3 previously defined. Applying the involution to these surfaces we get

$$H_i \sim (I)H_iR \qquad (i = 1, 2, 3),$$

where

$$R_{8n+16}y_3^2 = L^2 + LK(H_1y_4 - H_3y_2) - K^2H_1H_3^2 . \tag{11}$$

It is evident that $r \sim (I)R$. The order of R, as indicated, is $8n + 16$.

The transformation I applied to an F and an F' of $| F |$ and $| F' |$, respectively, gives

$$U \sim (I)R^nUG, \qquad U' \sim (I)R^nU'G, \tag{12}$$

where

$$G_{8n+16}y_3^2 = L^2 + 2K(VW' - V'W) \tag{13}$$

and $g \sim (I)G$.

Similarly

$$K \sim (I)R^{2n+2}GK, \qquad G \sim (I)R^{4n+8}G, \qquad R \sim (I)R^{4n+7}G^2. \tag{14}$$

Through a point O_r on r there is a quadric cone of lines of the congruence. Also through O_r passes one F of $| F |$ which meets the cone twice along r and in a residual curve k of order $2n + 4$. As O_r describes r the curve k generates the surface R, image of r under I.

Through a point O_g on g there is a unique line t of the congruence. However, every F of $| F |$ contains O_g since it lies upon the base curve of the pencil, hence $O_g \sim (I)t$. The ruled surface G, generated by t as O_g describes g, is the image of g under the involution.

Let us designate a point common to r and g as O_{rg}. The image of such a point, since it lies on r, is a curve k_{2n+4}. Furthermore, since the point also lies on g, the k_{2n+4} must contain a line l, hence is composite. The number of such lines l, that is, the number of intersections of r and g, is shown by the intersection of R and G to be $2n^2 + 12n$.

5. **Contact along fundamental curves.** In an involution, if a point O lies on the pointwise invariant surface K, either O is an isolated fundamental point whose image surface touches K, or O is a point of a fundamental curve w and K contains the whole of w and touches the corresponding image surface along w; and K cannot meet w in a general point [1; 180]. From (14) the fundamental curve g lies once upon the invariant surface K, hence K touches the image surface G along g and simple tangency exists. Similarly, r lying $2n + 2$ times on K, $2n + 2$ sheets of R touch K along this curve giving contact of order $2n + 2$.

6. **Parasitic lines.** Any line p of the congruence which meets g twice will be such that each point of the line will be carried into the entire line. If O_{gg} is an arbitrary point on such a line the surface F through O_{gg} is met by the line at that point, at $2n$ points on r and at the two points in which the line meets g, a total of $2n + 3$ points. It follows that the entire line lies upon F and that every point

of the line is the image of O_{gg}. Such a line is said to be parasitic and in the case of the present involution is seen from (14) to be double upon R and G, single upon K, and from (15) to lie once upon each homaloidal surface ϕ. The number of parasitic lines, as determined by the intersection of two homaloidal surfaces, is $3n^2 + 12n + 20$.

7. **Invariant and homaloidal surfaces.** An examination of equations (10) shows $K = 0$ to be pointwise invariant under the involution I. It may be noted that the surface $H_2^2 - 4H_1H_3 = 0$, the tangent developable of r, is invariant, but not pointwise invariant, under I.

Applying the involution to a generic plane π gives

$$\pi \equiv (Ax) \sim (I)L(A\ ) + K(AH) = y_3\phi,$$

where

$$(Ax) = \sum_{i=1}^{4} A_i x_i\,, \qquad (AH) = A_1H_1H_2 + A_2H_1H_3 - A_4H_3^2\,.$$

The ϕ's are homaloidal surfaces of the transformation and are of order $4n + 9$. Further

(15) $$\phi_{4n+9} \sim (I)(Ax)R^{2n+4}G;$$

hence, the homaloidal web is

$$\infty^3 \mid \phi \mid : r^{2n+4}g(3n^2 + 12n + 20)p,$$

where $(3n^2 + 12n + 20)p$ denotes the $3n^2 + 12n + 20$ parasitic lines. The intersection of two homaloidal surfaces,

$$H \equiv [\phi\phi]: r^{4n^2+16n+16}g(3n^2 + 12n + 20)pc_{4n+9}\,,$$

gives, along with the fundamental curves and the parasitic lines, a residual curve c_{4n+9} which is the image of the line $[\pi\pi]$. The totality of such curves c upon one ϕ produced by its intersection with the other ϕ's of the homaloidal web constitutes the homaloidal net of curves upon that ϕ, and is the image of the net of lines upon that plane π which is carried into ϕ by the transformation.

The Jacobian of the involution is $J_{16n+32} = RG$.

8. **Table of characteristics for the involution.** The images of planes and of fundamental elements may now be expressed by the following table:

$$\pi \sim (I)\phi: r^{2n+4}g(3n^2 + 12n + 20)p$$

$$r \sim (I)R: r^{4n+7+(2n+2)t}g^2 2(3n^2 + 12n + 20)p$$

$$g \sim (I)G: r^{4n+8}g^{1+1t}2(3n^2 + 12n + 20)p$$

$$K \sim (I)K: r^{2n+2+(2n+2)t}g^{1+1t}(3n^2 + 12n + 20)p,$$

where the coefficients of t in the multiplicities of r and g indicate the order of contact as found in §5.

9. **Intersection table.** The complete intersection table for the involution may now be written as follows:

$$[\phi\phi]: r^{4n^2+16n+16} g(3n^2 + 12n + 20)pc_{4n+9}$$

$$[\phi R]: r^{8n^2+30n+28} g^2 2(3n^2 + 12n + 20)pk_{2n+4,1}k_{2n+4,2}k_{2n+4,3}$$

$$[\phi G]: r^{8n^2+32n+32} g2(3n^2 + 12n + 20)pl_{1,1} \cdots l_{1,n^2+8n+4}$$

$$[\phi K]: r^{4n^2+12n+8} g(3n^2 + 12n + 20)pc_{4n+6}$$

$$[RG]: r^{16n^2+60n+56} g^2 4(3n^2 + 12n + 20)p(2n^2 + 12n)l$$

$$[RK]: r^{8n^2+22n+14} g^2 2(3n^2 + 12n + 20)p$$

$$[GK]: r^{8n^2+24n+16} g^{1+1t} 2(3n^2 + 12n + 20)p.$$

The three curves $k_{2n+4,1}$, $k_{2n+4,2}$, $k_{2n+4,3}$ of order $2n + 4$ are the images of the three points of intersection of π and r while the $n^2 + 8n + 4$ lines $l_{1,1}$, $\cdots$, l_{1,n^2+8n+4} are the images of the points of intersection of π and g. The curve c_{4n+6} is the intersection $[\pi K]$ and is invariant. The c_{4n+9} and the $2n^2 + 12n$ lines l have been explained in §§7 and 4, respectively.

10. **Definition of the non-involutorial transformation.** Consider a space cubic r and two projectively related pencils of surfaces

$$| F_{2n+1} |: r^n g_{n^2+4n+1} , \qquad | F'_{2n'+1} |: r^{n'} g'_{n'^2+4n'+1} .$$

Through a generic point $P(y)$ there passes a single F of $| F |$. The unique line t, belonging to the congruence and passing through $P(y)$, meets the associated F' of $| F' |$ in one residual point $P'(x)$, the image of $P(y)$ under the transformation so defined. The base curves of the pencils $| F |$ and $| F' |$ are denoted by g and g', respectively, and are of orders $n^2 + 4n + 1$ and $n'^2 + 4n' + 1$. Through any point $O_{g'}$ of g' there is a unique line t of the congruence, this line lying entirely upon one F' of $| F' |$ (see §12). The associated F meets t in one point $\overline{P}$ which generates a curve $\overline{g}$ as $O_{g'}$ describes g'. Similarly, beginning with a point O_g on g we find a point $\overline{P}'$ generating a curve $\overline{g}'$. It will be shown that r, g, g', $\overline{g}$ and $\overline{g}'$ are fundamental curves of the transformation.

11. **Equations of the transformation.** We again take the equation of r as

$$x_1 : x_2 : x_3 : x_4 = h^3 : h^2 : h : 1 \tag{16}$$

and the pencils of surfaces $| F |$ and $| F' |$ as

$$| F | \equiv U - u\overline{U} = 0, \qquad | F' | \equiv U' - u\overline{U}' = 0, \tag{17}$$

where

$$
\begin{aligned}
U_{2n+1} &= (ax)H_1^n + (bx)H_2^n + (cx)H_3^n \\
\overline{U}_{2n+1} &= (\bar{a}x)H_1^n + (\bar{b}x)H_2^n + (\bar{c}x)H_3^n \\
U'_{2n'+1} &= (a'x)H_1^{n'} + (b'x)H_2^{n'} + (c'x)H_3^{n'} \qquad (18) \\
(ax) &= \sum_{i=1}^{4} a_i x_i \\
H_1 &= x_1x_3 - x_2^2 , \qquad H_2 = x_1x_4 - x_2x_3 , \qquad H_3 = x_2x_4 - x_3^2 ,
\end{aligned}
$$

and so on.

Through a generic point $P(y)$ there passes one F of $|F|$ with parameter $u = U(y)/\overline{U}(y)$ and to this corresponds the F' of $|F'|$ whose equation is

$$
U'(x)\overline{U}(y) - \overline{U}'(x)U(y) = 0. \qquad (19)
$$

The unique line t of the congruence through $P(y)$ meets the cubic r in two points

$$
(h_1^3 , h_1^2 , h_1 , 1), \qquad (h_2^3 , h_2^2 , h_2 , 1), \qquad (20)
$$

where h_1 and h_2 are the roots of the quadratic $H_3h^2 - H_2h + H_1 = 0$ as before. The line through the points (20) meets the plane $x_3 = 0$ in the point

$$
(H_1H_2 , H_1H_3 , 0, -H_3^2) \qquad (21)
$$

and we again avoid irrational expressions by defining t as the line through $P(y)$ and the point (21). This line meets the surface (19) in one residual point $P'(x)$ having coordinates

$$
\begin{aligned}
\sigma x_1 &= Ly_1 + KH_1H_2 \\
\sigma x_2 &= Ly_2 + KH_1H_3 \\
\sigma x_3 &= Ly_3 \qquad (22) \\
\sigma x_4 &= Ly_4 - KH_3^2 ,
\end{aligned}
$$

where

$$
\begin{aligned}
L_{2n+2n'+5} &= U\overline{W}' - \overline{U}W', \qquad K_{2n+2n'+2} = U'\overline{U} - \overline{U}'U \\
W'_{2n'+4} &= [a'H]H_1^{n'} + [b'H]H_2^{n'} + [c'H]H_3^{n'} \qquad (23) \\
\overline{W}'_{2n'+4} &= [\bar{a}'H]H_1^{n'} + [\bar{b}'H]H_2^{n'} + [\bar{c}'H]H_3^{n'} \\
[a'H] &= a_1'H_1H_2 + a_2'H_1H_3 - a_4'H_3^2 ,
\end{aligned}
$$

and so on.

In a similar fashion the point $P'(x)$ determines an F' of $|F'|$ and a unique line of the congruence which intersects the corresponding F of $|F|$ in the point whose coordinates are given by

$$(24)\qquad \begin{aligned} \tau y_1 &= L'x_1 + K'H_1H_2 \\ \tau y_2 &= L'x_2 + K'H_1H_3 \\ \tau y_3 &= L'x_3 \\ \tau y_4 &= L'x_4 - K'H_3^2 , \end{aligned}$$

where

$$(25)\qquad \begin{aligned} L'_{2n+2n'+5} &= U'\overline{W} - \overline{U}'W, \qquad K'_{2n+2n'+2} = U\overline{U}' - U'\overline{U} = -K \\ W_{2n+4} &= [aH]H_1^n + [bH]H_2^n + [cH]H_3^n \\ \overline{W}_{2n+4} &= [\bar{a}H]H_1^n + [\bar{b}H]H_2^n + [\bar{c}H]H_3^n \\ [aH] &= a_1H_1H_2 + a_2H_1H_3 - a_4H_3^2 , \end{aligned}$$

and so on.

The point (21) being an ordinary point of the transformation and its inverse, y_3 is a factor of the right members of (22) and x_3 is a factor of the right members of (24). When these factors are discarded, equations (22) become those of the inverse transformation T^{-1} and (24) those of the direct transformation T, both transformations being of order $2n + 2n' + 5$. It may be pointed out that y_3 is not a factor of L or of K individually and, similarly, x_3 is not a factor of L' or of K' individually.

12. **Images of fundamental curves.** When the inverse transformation T^{-1} is applied to the quadrics H_1, H_2, H_3, previously defined, we find that

$$(26)\qquad H_i \sim (T^{-1})H_iR \qquad (i = 1, 2, 3),$$

where

$$(27)\qquad R_{4n+4n'+8}y_3^2 = L^2 + LK(H_1y_4 - H_3y_2) - K^2H_1H_3^2 .$$

Since the cubic r is the intersection of H_1, H_2 and H_3 it is evident that $r \sim (T^{-1})R$.

Similarly,

$$(28)\qquad H_i \sim (T)H_iR' \qquad (i = 1, 2, 3),$$

where

$$(29)\qquad R'_{4n+4n'+8}x_3^2 = L'^2 + L'K'(H_1x_4 - H_3x_2) - K'^2H_1H_3^2$$

and $r \sim (T)R'$.

The transformations T^{-1} and T applied to an F' and an F of $|F'|$ and $|F|$, respectively, give

$$(30)\qquad U' \sim (T^{-1})UR^{n'}G, \qquad U \sim (T)U'R'^{n}G',$$

where

$$G_{4n'+4}y_3 = U'\overline{W'} - \overline{U'}W', \qquad G'_{4n+4}x = U\overline{W} - \overline{U}W. \tag{31}$$

Here U and U' are corresponding surfaces of $|F|$ and $|F'|$ while g and g' are the residual base curves of $|F|$ and $|F'|$. It follows that $g' \sim (T^{-1})G$ and $g \sim (T)G'$.

Similarly,

$$K' \sim (T^{-1})K'R^{n+n'}GG', \qquad K \sim (T)KR'^{n+n'}GG'$$

$$K' \sim (T)K'R'^{n+n'}GG', \qquad K \sim (T^{-1})KR^{n+n'}GG'$$

$$G' \sim (T^{-1})R^{2n+2}G', \qquad G \sim (T)R'^{2n'+2}G$$

$$G' \sim (T)R'^{2n+2}G', \qquad G \sim (T^{-1})R^{2n'+2}G$$

$$R' \sim (T^{-1})R^{2n+2n'+3}G^2G'^2, \qquad R \sim (T)R'^{2n+2n'+3}G^2G'^2 \tag{32}$$

$$R' \sim (T) \cdot \frac{R'^{2n+2n'+3}[R'M'^2 + M'GG'(R'x_3^2 - L'^2 - K'^2H_1H_3^2) - K'^2G^2G'^2H_1H_3^2x_3^2]}{L'^2x_3^2}$$

$$R \sim (T^{-1}) \cdot \frac{R^{2n+2n'+3}[RM^2 + MGG'(Ry_3^2 - L^2 - K^2H_1H_3^2) - K^2G^2G'^2H_1H_3^2y_3^2]}{L^2y_3^2},$$

where

$$M'_{4n+4n'+10} = L'^2 + K'(W'\overline{W} - W\overline{W'}),$$

$$M_{4n+4n'+10} = L^2 + K(W\overline{W'} - \overline{W}W').$$

Through a point O_r on r there is a quadric cone of lines of the congruence. Associated with each generator (direction) of this cone is an F' of $|F'|$ and the corresponding F cuts the line in one residual point. The locus of such points is a curve c which generates the surface R, image of r under T^{-1} as O_r describes r. The order of c, determined by the intersection of R and a homaloidal surface, is $n + n' + 2$. R', the image of r under T, is generated in an analogous fashion.

Through a point $O_{g'}$ on g', there is a unique line t of the congruence. However, every F of $|F|$ is associated with $O_{g'}$, hence $O_{g'} \sim (T^{-1})t$. The ruled surface G generated by t as $O_{g'}$ describes g' is the image of g' under T^{-1}. Now consider the image of $O_{g'}$ under T. Through the point there is a unique line t of the congruence and one surface F of $|F|$. The line t meets the associated F' in a point $P'(x)$ which is the required image. However, F' has been met by t once at P', once at $O_{g'}$ (since every surface of $|F'|$ contains g') and $2n'$ times on r, a total of $2n' + 2$ intersections, hence the line t lies upon this F'. The associated F meets t in a residual point $\overline{P}$, hence $\overline{P} \sim (T)t$. The locus of points $\overline{P}$ is the curve $\overline{g}$ and $\overline{g} \sim (T)G$. The order $\overline{g}$, determined by the intersection of two homaloidal surfaces, is $n'^2 + 2nn' + 2n + 6n' + 7$. In a similar manner there is generated a ruled surface G' such that $g \sim (T)G'$, and a curve $\bar{g}'$ of order $n^2 + 2nn' + 2n' + 6n + 7$ such that $\bar{g}' \sim (T^{-1})G'$.

The image of a point O_{rg}, common to r and g, is a $c_{n+n'+2}$ since it lies on r. However, since the point also lies on g, the $c_{n+n'+2}$ must contain a line l, hence is composite. The number of such lines l, that is, the number of intersections of r and g, is shown by the intersection of R and G to be $2n^2 + 6n$.

13. **Invariant surfaces.** The surface $K = -K'$ is pointwise invariant under both T and T^{-1} as may be verified by noting equations (22) and (24) and the tangent developable of the cubic, $H_2^2 - 4H_1H_3$, is invariant, but not pointwise invariant, under the transformation.

14. **Tangency along** r. Let F and F' be corresponding surfaces of the pencils $| F |$ and $| F' |$, respectively, and k their intersection residual to r. The totality of such curves k is the pointwise invariant surface K' (or K) mentioned in the previous paragraph. Further, let O_{rk} be a point common to k and the cubic r which is known from (32) to lie upon both K and R'. As previously pointed out, there is a quadric cone of transversals of the cubic through this point and to each of these corresponds an F of $| F |$. The corresponding F' meets the transversal in a residual point, the totality of such points being a curve c which generates the surface R', image of r under T. At the point O_{rk} the curve c is tangent to that one of the generators of the quadric cone whose direction is the "invariant direction" [1; 175] through the point, that is, that generator which is tangent to k at O_{rk}. Since c generates R' and k generates K' the generator in question is tangent to both R' and K'. Furthermore, since r lies on both R' and K' the line through O_{rk} which is tangent to r is also tangent to both R' and K'. Accordingly, the curves r and k being distinct, the tangent planes to these surfaces at O_{rk} are coincident and R' and K' are tangent along r. In particular we see from (32) that r lies $(n + n')$-times upon K so that R' and K' have tangency of order $n + n'$ along r.

15. **Homaloidal surfaces.** Generic planes subjected to the transformations give

$$(33) \qquad \pi' \equiv (A'x) \sim (T^{-1})\phi, \qquad \pi \equiv (Ax) \sim (T)\phi,$$

where

$$(34) \qquad \begin{aligned} \phi_{2n+2n'+5}y_3 &= L(A'y) + K[A'H], \\ \phi'_{2n+2n'+5}x_3 &= L'(Ax) + K'[AH], \\ (A'x) &= \sum_{i=1}^{4} A'_i x_i , \end{aligned}$$

and so on. The ϕ's are homaloidal surfaces of the transformation. Further,

$$(35) \qquad \phi \sim (T)(A'x)R'^{n+n'+2}GG', \qquad \phi' \sim (T^{-1})(Ay)R^{n+n'+2}GG'$$

so that the homaloidal webs are

$$\infty^3 \mid \phi \mid : r^{n+n'+2} g\bar{g}, \qquad \infty^3 \mid \phi' \mid : r^{n+n'+2} g'\bar{g}'.$$

The intersections of pairs of homaloidal surfaces give

$$(36) \qquad \begin{aligned} [\phi\phi] &: r^{n^2+n'^2+2nn'+4n+4n'+4} g\bar{g} c_{2n+2n'+5} \\ [\phi'\phi'] &: r^{n^2+n'^2+2nn'+4n+4n'+4} g'\bar{g}' c'_{2n+2n'+5}, \end{aligned}$$

where $c_{2n+2n'+5}$ and $c'_{2n+2n'+5}$ are the images of the lines $[\pi'\pi']$ and $[\pi\pi]$ under T^{-1} and T, respectively. The c and c' may be thought of as generators of the homaloidal nets upon ϕ and ϕ', respectively, in the manner described in §7.

The Jacobian of the transformation is $J_{8n+8n'+16} = RGG'$.

16. **Table of characteristics.** The images of planes and of the fundamental elements under the non-involutorial transformation and its inverse can now be written

$$(37) \qquad \begin{aligned} & r \sim (T)R' : r^{2n+2n'+3+(n+n')t} g'^2 \bar{g}'^2 \\ & r \sim (T^{-1})R : r^{2n+2n'+3+(n+n')t'} g^2 \bar{g}^2 \\ & g \sim (T)G' : r^{2n+2} g\bar{g}', \qquad g' \sim (T^{-1})G : r^{2n'+2} g'\bar{g} \\ & \bar{g} \sim (T)G : r^{2n'+2} g'\bar{g}, \qquad \bar{g}' \sim (T^{-1})G' : r^{2n+2} g\bar{g}' \\ & \pi \sim (T)'\phi' : r^{n+n'+2} g'\bar{g}', \qquad \pi \sim (T^{-1})\phi : r^{n+n'+2} g\bar{g} \\ & K \sim (T)K' : r^{n+n'+(n+n')t} g\bar{g}g'\bar{g}' \\ & K' \sim (T^{-1})K : r^{n+n'+(n+n')t} g\bar{g}g'\bar{g}', \end{aligned}$$

where the coefficients of t in the multiplicities of r indicate the order of contact as found in §14.

17. **Intersection table.** The intersection table for the non-involutorial transformation follows:

$$\begin{aligned} [\phi'\phi'] &: r^{n^2+n'^2+2nn'+4n+4n'+4} g'\bar{g}' c_{2n+2n'+5} \\ [\phi'R'] &: r^{2n^2+2n'^2+4nn'+7n+7n'+6} g'^2 \bar{g}'^2 c_{n+n'+2,1} \cdots c_{n+n'+2,3} \\ [\phi'G'] &: r^{2n^2+2nn'+6n+2n'+4} \bar{g}' l_{1,1} \cdots l_{1,n^2+4n+1} \\ [\phi'K'] &: r^{n^2+2nn'+n'^2+2n+2n'} g'\bar{g}' k_{2n+2n'+2} \\ [R'G'] &: r^{4n^2+4nn'+10n+4n'+6} \bar{g}'^2 c_{1,1} \cdots c_{1,2n^2+6n} \\ [R'K'] &: r^{2n^2+4nn'+2n'^2+3n+3n'+(n+n')t} g'^2 \bar{g}'^2 \\ [G'K'] &: r^{2n^2+2nn'+2n+2n'} g\bar{g}'. \end{aligned}$$

The $c_{2n+2n'+5}$ has been defined in §15. The $c_{n+n'+2,1} \cdots c_{n+n'+2,3}$ in $[\phi'R']$ are the images of π and r. The lines $l_{1,1} \cdots l_{1,n^2+4n+1}$ in $[\phi'G']$ are the images of the $n^2 + 4n + 1$ intersections of π and g while the $k_{2n+2n'+2}$ in $[\phi'K']$ is the intersection of K and π. The lines $c_{1,1} \cdots c_{1,2n^2+6n}$ in $[R'G']$ were explained in §12.

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