Transformation Groups
Vol. 20, No. 1, 2015, pp. 291–304

THE GENERALIZED AMALGAMATED PRODUCT STRUCTURE OF THE TAME AUTOMORPHISM GROUP IN DIMENSION THREE

DAVID WRIGHT

Department of Mathematics
Washington University
St. Louis, MO 63130, USA
wright@math.wustl.edu

Abstract. For K a field of characteristic zero, it is shown that the tame subgroup $\mathrm{TA}_3(K)$ of the group $\mathrm{GA}_3(K)$ of polynomial automorphisms of $\mathbb{A}^3_K$ can be realized as a generalized amalgamated product, specifically, the product of three subgroups, amalgamated along pairwise intersections, in a manner that generalizes the well-known amalgamated free product structure of $\mathrm{TA}_2(K)$ (which coincides with $\mathrm{GA}_2(K)$ by Jung's Theorem). The result follows from defining relations for $\mathrm{TA}_3(K)$ given by U. U. Umirbaev.

1. Polynomial automorphism groups

For a commutative ring R, we write $R^{[n]}$ for the polynomial ring $R[X_1, \ldots, X_n]$ in n variables over R. We will have numerous occasions in which we will refer to the subalgebra $R[X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n]$ for $i \in \{1, \ldots, n\}$, so we will use the shorter notation $R[X, \widehat{i}\,]$ to denote it.

The symbol $\mathrm{GA}_n(R)$ denotes the *general automorphism group,* by which we mean the automorphism group of $\mathrm{Spec} R^{[n]}$ over $\mathrm{Spec} R$. As such, it is anti-isomorphic to the group of R-algebra automorphisms of $R^{[n]}$. An element φ of $\mathrm{GA}_n(R)$ is represented by a vector $\varphi = (F_1, \ldots, F_n) \in (R^{[n]})^n$.

The general linear group $\mathrm{GL}_n(R)$ is contained in $\mathrm{GA}_n(R)$ in an obvious way. Another familiar subgroup is $\mathrm{EA}_n(R)$, the group generated by *elementary* automorphisms, i.e., those of the form $(X_1, \ldots, X_{i-1}, X_i + f, X_{i+1}, \ldots, X_n)$ for some $i \in \{1, \ldots, n\}$, $f \in R[X, \widehat{i}\,]$.

The subgroup of *tame automorphisms,* denoted $\mathrm{TA}_n(R)$, is the subgroup generated by $\mathrm{GL}_n(R)$ and $\mathrm{EA}_n(R)$. Another subgroup of interest is the *affine* group $\mathrm{Af}_n(R)$, which is the group generated by $\mathrm{GL}_n(R)$ together with the *translations,* i.e., those automorphisms of the form $(X_1 + a_1, \ldots, X_n + a_n)$ with $a_1, \ldots, a_n \in R$.

For K a field, $\mathrm{GA}_n(K)$ is sometimes called the *affine Cremona group.* It sits naturally as a subgroup of the full Cremona group $\mathrm{Cr}_n(K)$, which is the group of birational automorphisms of affine (or projective) n-space. The Jung–Van der Kulk Theorem ([6], [7]) states that $\mathrm{TA}_2(K) = \mathrm{GA}_2(K)$. Shestakov and Umirbaev

DOI: 10.1007/s00031-014-9290-9
Received November 27, 2013. Accepted April 23, 2014.
Published online November 23, 2014.
Corresponding Author: D. Wright, e-mail: wright@math.wustl.edu.

([11]) showed that $\mathrm{TA}_3(K) \neq \mathrm{GA}_3(K)$ when K has characteristic zero. This paper deals with the structure of $\mathrm{TA}_3(K)$, when $\mathrm{char}(K) = 0$, based of work on Umirbaev in [13].

2. Generalized amalgamated products of groups

We begin with the definition of a generalized amalgamation of groups. We refer the reader to [8] for a comprehensive discussion of combinatorial group theory.

Definition 1. Suppose we are given groups A_i for each $i \in \{1, \dots, n\}$, and for each $i, j \in \{1, \dots, n\}$ with $i \neq j$ we have groups $B_{ij} = B_{ji}$ with injective homomorphisms $\varphi_{ij} : B_{ij} \to A_i$ which are compatible, meaning if i, j, k are distinct then $\varphi_{ij}^{-1}(\varphi_{ik}(B_{ik})) = \varphi_{ji}^{-1}(\varphi_{jk}(B_{jk}))$ and on this group $\varphi_{ik}^{-1}\varphi_{ij} = \varphi_{jk}^{-1}\varphi_{ji}$. This gives set-theoretic gluing data by which we can compatibly glue A_i to A_j along B_{ij} via $\varphi_{ij}^{-1}\varphi_{ji}$ forming an amalgamated union S of the sets $A_1, \dots, A_n$. We then form the free group $\mathcal{F}$ on S, denoting the group operation on $\mathcal{F}$ by $*$. For $i \in \{1, \cdots, n\}$ and $x, y \in A_i \subset S$, we let $r_{x,y} = x * y * (xy)^{-1} \in \mathcal{F}$ (where xy is the product in A_i). Finally we let $\mathcal{G}$ be the quotient of $\mathcal{F}$ by all the relations $r_{x,y}$. The group $\mathcal{G}$ is called the *generalized amalgamated product* of the groups[2] A_i, $i \in \{1, \dots, n\}$ along the groups B_{ij}, $i, j \in \{1, \dots, n\}$. There are natural group homomorphisms $\iota_i : A_i \to \mathcal{G}$ with $\iota_i\varphi_{ij} = \iota_j\varphi_{ji}$ on B_{ij}.

The group $\mathcal{G}$ has the following universal property: Given a group H and maps $\rho_i : A_i \to H$ for $i \in \{1, \dots, n\}$ such that $\rho_i\varphi_{ij} = \rho_j\varphi_{ji}$ on B_{ij} for all $i, j \in \{1, \dots, n\}$, then there is a unique map $\Phi : \mathcal{G} \to H$ with $\rho_i = \Phi\iota_i$ for all i.

When there are only two subgroups A_1 and A_2 containing a common subgroup B, $\mathcal{G}$ is the usual amalgamated free product. In this case the two groups inject into the amalgamated product and a very strong factorization theorem holds. Moreover, the Bass–Serre tree theory of groups acting on trees (see [10]) provides a tree on which $\mathcal{G}$ acts without inversion, having a fundamental domain consisting of a single edge with its end vertices, the stabilizers of the vertices being A_1 and A_2 and the stabilizer of the edge the common subgroup B.

Such theorems do not hold in general for generalized amalgamations of three or more groups along pairwise intersections. The groups A_i may not map injectively into $\mathcal{G}$, and in fact $\mathcal{G}$ may be the trivial group when none of the groups A_i are trivial, as the following example from [12] shows.

Example 1. For $\{i, j, k\} = \{1, 2, 3\}$ let B_{ij} be the infinite cyclic group generated by b_k. Let

$$\begin{aligned} A_1 &= \langle b_2, b_3 | b_2 b_3 b_2^{-1} = b_3^2 \rangle, \\ A_2 &= \langle b_3, b_1 | b_3 b_1 b_3^{-1} = b_1^2 \rangle, \\ A_3 &= \langle b_1, b_2 | b_1 b_2 b_1^{-1} = b_2^2 \rangle. \end{aligned}$$

Then B_{ij} is a common subgroup of A_i and A_j and we can form the generalized amalgamation $\mathcal{G}$ of the groups A_i along the groups B_{ij}. It can be shown that in this case $\mathcal{G}$ is the trivial group.

[2]We use the term *generalized* amalgamated product to distinguish it from the usual amalgamated product of two or more groups along a single common subgroup.

Whether such amalgamation data gives rise to the group acting on a simplicial complex is not easy to detect (see, for example, [12], [5], and [2]). It occurs precisely when each of the groups A_{ij} maps injectively to $\mathcal{G}$, and in this situation, the amalgamated union S maps injectively to $\mathcal{G}$ as well. The n-simplex of groups arising from this data is called *developable* by Haefliger ([5]) in case of this occurence.

However, if the groups A_i are subgroups of a given group G and if we take B_{ij} to be $A_i \cap A_j$ and φ_{ij} the inclusion map within G, then clearly there exists a homomorphism $\Phi : \mathcal{G} \to G$ restricting to the identity on each A_i, which shows that in this case the amalgamated union S maps injectively to $\mathcal{G}$. The map Φ will be surjective precisely when G is generated by the subgroups $A_1, \ldots, A_n$. If Φ is an isomorphism, then the structure of $\mathcal{G}$ arises from the action of G on an n-dimensional simply connected simplicial complex, with a single simplex serving as a fundamental domain.

Automorphism groups of various kinds can be realized as generalized amalgamations of groups. Some examples are given below.

Example 2. $\mathrm{SL}_2(\mathbb{Z}) \simeq (\mathbb{Z}/4\mathbb{Z}) *_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}/6\mathbb{Z})$ acts on the upper half plane. The generator of $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$ can be taken to be $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, respectively. Here the translates of the circular arc $z = e^{i\theta}$ with $\pi/3 \leq \theta \leq \pi/2$ form a tree with this arc as a fundamental domain, and this is the tree given by the Bass–Serre theory.

Example 3 (Nagao's Theorem [9]**).** For K a field we have the amalgamated product $\mathrm{GL}_2(K[X]) = \mathrm{GL}_2(K) *_{\mathrm{B}_2(K)} \mathrm{B}_2(K[X])$, with B_2 denoting the lower triangular group. (Here X represents a singly variable.) This structure can be realized via the Bass–Serre theory by the action of $\mathrm{GL}_2(K[X])$ on a tree whose vertices are $\mathcal{O}$-lattices in the rank two vector space over $K(X)$, where $\mathcal{O}$ is the DVR of $K(X)$ with uniformizing parameter $1/X$ (see [10]). Here the fundamental domain is not just a single edge, but an edge connected to a "directed geodesic."

Example 4 (Jung–Van der Kulk Theorem [6],[7]**).** For K a field, the group $\mathrm{GA}_2(K)$ of polynomial automorphisms of the affine plane has the amalgamated product structure $\mathrm{GA}_2(K)) = \mathrm{Af}_2(K) *_{\mathrm{Bf}_2(K)} \mathrm{BA}_2(K)$. Here BA_2 is the group of automorphisms of the form $(\alpha X_1 + \gamma, \beta X_2 + f(X_1))$, $\alpha, \beta, \gamma \in K$, $\alpha\beta \neq 0$, $f(X_1) \in K[X_1]$, and $\mathrm{Bf}_2(K) = \mathrm{Af}_2(K) \cap \mathrm{BA}_2(K)$. Again, this structure arises from the action of $\mathrm{GA}_2(K)$ on a tree whose vertices are certain complete algebraic surfaces realized as collections of local rings ("models") inside the function field $K(X_1, X_2)$ (see [14]).

Remark 1. One might wonder if $\mathrm{TA}_3(K)$ is the amalgamated product of $\mathrm{Af}_3(K)$ with the triangular group $\mathrm{BA}_3(K)$, the latter consisting of automorphisms of the form $(\alpha X_1 + \gamma, \beta X_2 + f(X_1), \kappa X_3 + g(X_1, X_2))$, $\alpha, \beta, \kappa, \gamma \in K$, $\alpha\beta\kappa \neq 0$, $f(X_1) \in K[X_1]$, $g(X_1, X_2) \in K[X_1, X_2]$. This is known not to be the case.

Example 5. The full Cremona group $\mathrm{Cr}_2(K)$ over an algebraically closed field K is the generalized amalgamation of three groups: the automorphism group of $\mathbb{P}^2_K$ (which is $\mathrm{PGL}_2(K)$), the automorphism group of $\mathbb{P}^1_K \times \mathbb{P}^1_K$, and thirdly the K-automorphism group of $\mathbb{P}^1_L$ where $L = K(t)$, with t transcendental over K.

There is a naturally realizable simplicial complex of triangles $\mathcal{C}$ on which $\mathrm{Cr}_2(K)$ acts which yields this structure and also contains the tree of Example 4 with the action of $\mathrm{GA}_2(K)$ being the restriction of the action of $\mathrm{Cr}_2(K)$ on $\mathcal{C}$. See [14] for details.

Remark 2. It has been shown by Stéphane Lamy (unpublished) that the complex $\mathcal{C}$ in Example 5 is not 2-connected.

3. Polynomial automorphisms in dimension three

A major breakthrough came in 2004 when Shestakov and Umirbaev showed that the automorphism group $\mathrm{GA}_3(K)$ properly contains the tame subgroup $\mathrm{TA}_3(K)$ when K is a field of characteristic zero ([11]). Specifically, they showed that the automorphism

$$\left(X_1 + X_3(X_2X_3 + X_1^2), X_2 - 2X_1(X_2X_3 + X_1^2) - X_1(X_2X_3 + X_1^2)^2, X_3\right)$$

is not tame, resolving a conjecture of Nagata from 1972. The group $\mathrm{GA}_3(K)$ remains a mystery, as no describable set of generators has been given.

However, the tame subgroup $\mathrm{TA}_3(K)$ is (by definition) generated by the elementary and linear automorphisms, which are familiar. Moreover, a set of generating relations has been given by Umirbaev in [13]. This paper will show that a generalized amalgamated product structure for $\mathrm{TA}_3(K)$ results from Umirbaev's relations. We begin by presenting those results.

For $\varphi = (F_1, \ldots, F_n) \in \mathrm{GA}_n(K)$ and $f \in K^{[n]}$ we write $f(\varphi)$ for $f(F_1, \ldots, F_n)$. This defines an action (on the right) of $\mathrm{GA}_n(K)$ on $K^{[n]}$.

For $i \in \{1, \ldots, n\}$, $\alpha \in K$, and $f \in K[X, \widehat{i}\,]$, consider the automorphism

$$\sigma_{i,\alpha,f} = (X_1, \ldots, X_{i-1}, \alpha X_i + f, X_{i+1}, \ldots, X_n), \tag{1}$$

which is easily seen to be tame. (Note that, although $\sigma_{i,\alpha,f}$ depends on n, the n is suppressed in order to simplify notation, and will be understood by context.) Given $k, \ell \in \{1, \ldots, n\}$, $k \neq \ell$, we define a tame automorphism $\tau_{k.\ell}$ by

$$\tau_{k.\ell} = \sigma_{\ell,1,X_k} \sigma_{k,1,-X_\ell} \sigma_{\ell,-1,X_k} \tag{2}$$

A simple calculation shows that $\tau_{k,\ell}$ is the transposition switching the X_k and X_ℓ coordinates.

One can check directly that

$$\sigma_{i,\alpha,f} \sigma_{i,\beta,g} = \sigma_{i,\alpha\beta,f+\alpha g}. \tag{3}$$

Also, if $i, j \in \{1, \ldots, n\}$, $i \neq j$, and if $f \in K[X, \widehat{i}\,] \cap K[X, \widehat{j}\,]$, $g \in K[X, \widehat{j}\,]$, then

$$\sigma_{i,\alpha,f}{}^{-1} \sigma_{j,\beta,g} \sigma_{i,\alpha,f} = \sigma_{j,\beta,g(\sigma_{i,\alpha,f})}. \tag{4}$$

It follows that if $g \in K[X, \widehat{i}\,] \cap K[X, \widehat{j}\,]$ then $\sigma_{i,\alpha,f}$ and $\sigma_{i,\beta,g}$ commute.

Let $k, \ell \in \{1, \ldots, n\}$, $k \neq \ell$. For $i \in \{1, \ldots, n\}$ let j be the image of i under the permutation which switches k and ℓ; in other words, the element of $\{1, \ldots, n\}$ for which $X_j = X_i(\tau_{k,\ell})$. Then we have

$$\tau_{k,\ell} \sigma_{i,\alpha,f} \tau_{k,\ell} = \sigma_{j,\alpha,f(\tau_{k,\ell})}. \tag{5}$$

Theorem 4.1 of [13] asserts the following.

Theorem 1 (Umirbaev). *Let K be a field of characteristic zero. The relations* (3), (4), *and* (5) *are defining relations for $TA_3(K)$ with respect to the generators $\sigma_{i,\alpha,f}$ defined in* (1). *Here $\tau_{k,\ell}$ in* (5) *is defined formally in terms of these generators by* (2).

This will be the key tool in the proof of Theorem 2, which is the main result of this paper.

4. Subgroups of interest

For $i \in \{1, \ldots, n\}$, let V_i be the sub-vector space of $K^{[n]}$ generated by K and the variables $X_1, \ldots, X_i$, i.e.,

$$V_i = K \oplus KX_1 \oplus \cdots \oplus KX_i. \tag{6}$$

Then H_i is defined to be the stabilizer of V_i in $\mathrm{GA}_n(K)$ via the action defined in Section 3, i.e.,

$$H_i = \{\varphi \in \mathrm{GA}_n(K) \mid \varphi(V_i) = V_i\}. \tag{7}$$

Note that H_n is the affine group $\mathrm{Af}_n(K)$. More generally, the subgroup of H_i that fixes each of the variables $X_{i+1}, \ldots, X_n$ can be identified with $\mathrm{Af}_i(K)$. In fact, H_i retracts onto $\mathrm{Af}_i(K)$ via the map $\varphi = (F_1, \ldots, F_n) \mapsto (F_1, \ldots, F_i)$, and the kernel of this retraction is the subgroup of H_i consisting of the elements that fix each of the variables $X_1, \ldots, X_i$, which is $\mathrm{GA}_{n-i}(K[X_1, \ldots, X_i])$. Thus H_i has the semidirect product structure

$$H_i = \mathrm{Af}_i(K) \ltimes \mathrm{GA}_{n-i}(K[X_1, \ldots, X_i]) \tag{8}$$

(where, for $i = n$, we read this as $H_n = \mathrm{Af}_n(K)$). These subgroups are defined in [3, p. 23], where it is conjectured that together they generate $\mathrm{GA}_n(K)$ (Conjecture 14.1) and that (whether or not that conjecture is true) the subgroup generated by $H_1, \ldots, H_n$ is the generalized amalgamated product of these groups along pairwise intersections (Conjecture 14.2). It should be noted that Freudenburg produced an example (see [4, p. 171]) of an automorphism in $\mathrm{GA}_3(K)$ which has not been shown to lie in this subgroup.[3]

Furthermore, the groups $\widetilde{H}_i$ are defined by

$$\widetilde{H}_i = H_i \cap \mathrm{TA}_n(K), \tag{9}$$

which are easily seen to generate $\mathrm{TA}_n(K)$. We can surmise from (8) that

$$\widetilde{H}_i \supseteq \mathrm{Af}_i(K) \ltimes \mathrm{TA}_{n-i}(K[X_1, \ldots, X_i]). \tag{10}$$

For $i = n$ equality holds trivially and we have $\widetilde{H}_n = H_n$, both being equal to $\mathrm{Af}_n(K)$. For $i = n - 1$ it is also easily seen that equality holds in (10) and

[3] This example is also of interest because it has not been shown to be *stably tame.* (See [1] for the definition of this concept.)

moreover we have $\widetilde{H}_{n-1} = H_{n-1}$ since TA_1 and GA_1 coincide over an integral domain (even a reduced ring).

There is one other case where the containment of (10) is known to be an equality. Namely, for $n = 3$ and K of characteristic zero we have

$$\widetilde{H}_1 = \mathrm{Af}_1(K) \ltimes \mathrm{TA}_2(K[X_1]). \tag{11}$$

This follows from [11, Cor. 10], a very deep result asserting that in $\mathrm{GA}_3(K)$ we have

$$\mathrm{GA}_2(K[X_1]) \cap \mathrm{TA}_3(K) = \mathrm{TA}_2(K[X_1]).$$

This together with the known proper containment $\mathrm{TA}_2(K[X_1]) \subsetneqq \mathrm{GA}_2(K[X_1])$ tells us that $\widetilde{H}_1 \subsetneqq H_1$ for $n = 3$. It is not known whether $\widetilde{H}_1 \subsetneqq H_1$ when $n > 3$.

It is conjectured that $\mathrm{TA}_n(K)$ is the generalized amalgamated product of the subgroups $\widetilde{H}_1, \ldots, \widetilde{H}_n$ along pairwise intersections ([3, Conj. 14.3]). The main result of this paper is that this conjecture is true for $n = 3$ and K a field of characteristic zero. In light of the above observations, for $n = 3$ we have $\widetilde{H}_2 = H_2$ and $\widetilde{H}_3 = H_3$ (but *not* $\widetilde{H}_1 = H_1$), so this can be stated as:

Theorem 2. *For K a field of characteristic zero,* $\mathrm{TA}_3(K)$ *is the generalized amalgamated product of the three groups $\widetilde{H}_1, H_2, H_3$ along their pairwise intersections.*

This will be proved in the next section.

5. Proof of Theorem 2

The main tool in the proof is Theorem 1, which asserts that $\mathrm{TA}_3(K)$ is generated by the elements $\sigma_{i,\alpha,f}$ as defined in (1) subject to the relations (3), (4), and (5).

Let $\mathcal{F}$ be the free group generated by the formal symbols $[\sigma_{i,\alpha,f}]$, with $i \in \{1, 2, 3\}$, $f \in K[X, \widehat{i}]$, $\alpha \in K$. Accordingly, we rewrite the relations (3), (4), and (5) replacing each σ by its corresponding formal symbol $[\sigma]$:

$$[\sigma_{i,\alpha,f}][\sigma_{i,\beta,g}] = [\sigma_{i,\alpha\beta,f+\alpha g}], \tag{R1}$$

$$[\sigma_{i,\alpha,f}]^{-1}[\sigma_{j,\beta,g}][\sigma_{i,\alpha,f}] = [\sigma_{j,\beta,g(\sigma_{i,\alpha,f})}], \tag{R2}$$

$$[\tau_{k,\ell}][\sigma_{i,\alpha,f}][\tau_{k,\ell}] = [\sigma_{j,\alpha,f(\tau_{k,\ell})}], \tag{R3}$$

where, in (R2), $i \neq j$, $f \in K[X, \widehat{i}] \cap K[X, \widehat{j}]$, $g \in K[X, \widehat{j}]$, and, in (R3), $k \neq \ell$, j is the image of i under the permutation which switches k and ℓ, and

$$[\tau_{k,\ell}] = [\sigma_{\ell,1,X_k}][\sigma_{k,1,-X_\ell}][\sigma_{\ell,-1,X_k}] \tag{12}$$

(after (2)). Let $\mathcal{N}$ be the normal subgroup of $\mathcal{F}$ generated by (R1), (R2), and (R3).

Theorem 1 says that the homomorphism from $\mathcal{F}$ to $\mathrm{TA}_3(K)$ sending $[\sigma_{i,\alpha,f}]$ to $\sigma_{i,\alpha,f}$ induces an isomorphism

$$\mathcal{F}/\mathcal{N} \xrightarrow{\cong} \mathrm{TA}_3(K). \tag{13}$$

Let $\mathfrak{G}$ be the generalized amalgamated product of $\widetilde{H}_1, H_2, H_3$ along their pairwise intersections. The inclusions of $\widetilde{H}_1, H_2, H_3$ in $\mathrm{TA}_3(K)$ induce a group homomorphism $\Phi : \mathfrak{G} \to \mathrm{TA}_3(K)$ which is surjective since the three subgroups generate $\mathrm{TA}_3(K)$ (in fact any two of them generate). We will define a group homomorphism from $\mathrm{TA}_3(K)$ to $\mathfrak{G}$ using the isomorphism (13) and show that it is inverse to Φ, thus proving the theorem.

We first define a homomorphism $\widehat{\Psi} : \mathcal{F} \to \mathfrak{G}$, which is accomplished by specifying the images of the free generators $[\sigma_{i,\alpha,f}]$. We will then show that $\mathcal{N}$ lies in the kernel of $\widehat{\Psi}$, thus inducing a map $\Psi : \mathcal{F}/\mathcal{N} = \mathrm{TA}_3(K) \to \mathfrak{G}$ (identifying $\mathcal{F}/\mathcal{N}$ and $\mathrm{TA}_3(K)$ via (13)), which will be shown to be the inverse of Φ.

According to the discussion in Section 2, $\mathfrak{G}$ contains the amalgamated union of $\widetilde{H}_1, H_2, H_3$, as does $TA_3(K)$, with Φ restricting to the identity map on this set. Let us denote by $\widetilde{\mathfrak{H}}_1, \mathfrak{H}_2, \mathfrak{H}_3$ the isomorphic copies of $\widetilde{H}_1, H_2, H_3$, respectively, that lie inside $\mathfrak{G}$. It is important to keep in mind that $\widetilde{\mathfrak{H}}_1 \cup \mathfrak{H}_2 \cup \mathfrak{H}_3$ maps bijectively to $\widetilde{H}_1 \cup H_2 \cup H_3$ via Φ.

Note that if $i = 2$ or $i = 3$ then $\sigma_{i,\alpha,f}$ lies in $\widetilde{H}_1$ and if $\deg f \leq 1$ then $\sigma_{i,\alpha,f}$ lies in H_3, so in each of these cases $\sigma_{i,\alpha,f}$ can be viewed as an element of the union $\widetilde{\mathfrak{H}}_1 \cup \mathfrak{H}_2 \cup \mathfrak{H}_3 \subset \mathfrak{G}$. To avoid confusion, we will denote these elements of $\mathfrak{G}$ by $\mathfrak{s}_{i,\alpha,f}$. Thus it makes sense to make the assignments

$$\widehat{\Psi}([\sigma_{i,\alpha,f}]) = \mathfrak{s}_{i,\alpha,f} \in \widetilde{\mathfrak{H}}_1 \quad \text{for } i = 2, 3, \tag{14}$$

$$\widehat{\Psi}([\sigma_{1,\alpha,f}]) = \mathfrak{s}_{1,\alpha,f} \in \mathfrak{H}_3 \quad \text{for } \deg f \leq 1. \tag{15}$$

Since the factors of (12) involve only polynomials of degree ≤ 1, $\widehat{\Psi}([\tau_{k,\ell}])$ is defined by applying $\widehat{\Psi}$ to those factors using (14) and(15) above. We will denote the resulting element of $\mathfrak{G}$ by $\mathfrak{t}_{k,\ell}$. Thus:

$$\widehat{\Psi}([\tau_{k,\ell}]) = \mathfrak{t}_{k,\ell} = \mathfrak{s}_{\ell,1,X_k}\mathfrak{s}_{k,1,-X_\ell}\mathfrak{s}_{\ell,-1,X_k}, \tag{16}$$

and this is just the permutation in $\mathfrak{H}_3 \cong \mathrm{Af}_3(K)$ that switches X_k and X_ℓ. It remains to define $\widehat{\Psi}([\sigma_{1,\alpha,f}])$ for arbitrary $f \in K[X_2, X_3]$. This we do as follows:

$$\widehat{\Psi}\left([\sigma_{1,\alpha,f}]\right) = \mathfrak{t}_{1,3}\mathfrak{s}_{3,\alpha,f(X_2,X_1)}\mathfrak{t}_{1,3}. \tag{17}$$

The reader will easily verify that this assignment coincides with (15) in the case $\deg f \leq 1$, since both occur in $\mathfrak{H}_3$.

Thus we have defined $\widehat{\Psi} : \mathcal{F} \to \mathfrak{G}$, and we must now show that the subgroup $\mathcal{N}$ lies in the kernel of $\widehat{\Psi}$, i.e, that equations (R1), (R2), and (R3) hold, replacing σ by $\mathfrak{s}$ and τ by $\mathfrak{t}$. This gets a bit tedious because of the asymmetry in the definitions of $\widehat{\Psi}\left([\sigma_{i,\alpha,f}]\right)$ depending on i.

We begin with (R1). Note that if $i = 2$ or $i = 3$, then according to (14), this amounts to showing that

$$\mathfrak{s}_{i,\alpha,f}\mathfrak{s}_{i,\beta,g} = \mathfrak{s}_{i,\alpha\beta,f+\alpha g} \quad \text{for } i = 2, 3. \tag{18}$$

But this is a relation that takes place in $\widetilde{\mathfrak{H}}_1$, so it holds in $\mathfrak{G}$. For $i = 1$ we must use (17). For $f, g \in K[X_2, X_3]$ we have

$$\begin{aligned}
\widehat{\Psi}([\sigma_{1,\alpha,f}])\widehat{\Psi}([\sigma_{1,\beta,g}]) &= (\mathfrak{t}_{1,3}\mathfrak{s}_{3,\alpha,f(X_2,X_1)}\mathfrak{t}_{1,3})(\mathfrak{t}_{1,3}\mathfrak{s}_{3,\beta,g(X_2,X_1)}\mathfrak{t}_{1,3}) \\
&= \mathfrak{t}_{1,3}\mathfrak{s}_{3,\alpha,f(X_2,X_1)}\mathfrak{s}_{3,\beta,g(X_2,X_1)}\mathfrak{t}_{1,3} \\
&\qquad\qquad\qquad\qquad (\text{since } \mathfrak{t}_{1,3}^2 = 1 \text{ in } \mathfrak{H}_3) \\
&= \mathfrak{t}_{1,3}\mathfrak{s}_{3,\alpha\beta,f+\alpha g}\mathfrak{t}_{1,3} \qquad \text{by (18)} \\
&= \widehat{\Psi}([\sigma_{1,\alpha\beta,f+\alpha g}]) \qquad \text{by (17)}
\end{aligned}$$

completing the proof that the relation (R1) is respected by $\widehat{\Psi}$.

We now address (R2). Recall that $i, j \in \{1, 2, 3\}$ and $i \neq j$. If $\{i, j\} = \{2, 3\}$ we must show, again appealing to (14), that

$$\mathfrak{s}_{i,\alpha,f}^{-1}\mathfrak{s}_{j,\beta,g}\mathfrak{s}_{i,\alpha,f} = \mathfrak{s}_{j\beta,g(\sigma_{i,\alpha,f})} \quad \text{for } i = 2, 3. \tag{19}$$

But, again, this is a relation that holds in $\widetilde{\mathfrak{H}}_1$, hence in $\mathfrak{G}$.

We now consider the case $i = 1$, $j = 3$. We will use the following basic permutation relation, which holds in the symmetric group $\mathfrak{S}_3 \subset \mathfrak{H}_3$ (hence it holds in $\mathfrak{G}$) for $\{k, \ell, m\} = \{1, 2, 3\}$:

$$\mathfrak{t}_{k,\ell} = \mathfrak{t}_{k,m}\mathfrak{t}_{m,\ell}\mathfrak{t}_{k,m}. \tag{20}$$

In the equations below the underbrace indicates what will be replaced in the next line; the overbrace in the next line marks the equivalent expression that has been substituted.

For $f \in K[X_2]$ and $g \in K[X_1, X_2]$,

$$\begin{aligned}
&\widehat{\Psi}\big([\sigma_{1,\alpha,f(X_2)}]^{-1}[\sigma_{3,\beta,g(X_1,X_2)}][\sigma_{1,\alpha,f(X_2)}]\big) \\
&\quad = \widehat{\Psi}\left([\sigma_{1,\alpha,f(X_2)}]\right)^{-1}\widehat{\Psi}\left([\sigma_{3,\beta,g(X_1,X_2)}]\right)\widehat{\Psi}\left([\sigma_{1,\alpha,f(X_2)}]\right) \\
&\quad = \Big(\underbrace{\mathfrak{t}_{1,3}}\,\mathfrak{s}_{3,\alpha,f(X_2)}{}^{-1}\,\underbrace{\mathfrak{t}_{1,3}}\Big)\big(\mathfrak{s}_{3,\beta,g(X_1,X_2)}\big)\Big(\underbrace{\mathfrak{t}_{1,3}}\,\mathfrak{s}_{3,\alpha,f(X_2)}\,\underbrace{\mathfrak{t}_{1,3}}\Big) \text{ by (14), (17)}
\end{aligned}$$

Applying (20) to $\mathfrak{t}_{1,3}$:

$$= \mathfrak{t}_{1,2}\mathfrak{t}_{2,3}\underbrace{\mathfrak{t}_{1,2}\mathfrak{s}_{3,\alpha,f(X_2)}{}^{-1}\mathfrak{t}_{1,2}}\mathfrak{t}_{2,3}\mathfrak{t}_{1,2}\mathfrak{s}_{3,\beta,g(X_1,X_2)}\mathfrak{t}_{1,2}\mathfrak{t}_{2,3}\underbrace{\mathfrak{t}_{1,2}\mathfrak{s}_{3,\alpha,f(X_2)}\mathfrak{t}_{1,2}}\mathfrak{t}_{2,3}\mathfrak{t}_{1,2}$$

Using the relation $\mathfrak{t}_{1,2}\mathfrak{s}_{3,\alpha,f(X_2)}\mathfrak{t}_{1,2} = \mathfrak{s}_{3,\alpha,f(X_1)}$ from $\mathfrak{H}_2$:

$$= \mathfrak{t}_{1,2}\mathfrak{t}_{2,3}\,\overbrace{\mathfrak{s}_{3,\alpha,f(X_1)}}{}^{-1}\,\mathfrak{t}_{2,3}\,\underbrace{\mathfrak{t}_{1,2}\mathfrak{s}_{3,\beta,g(X_1,X_2)}\mathfrak{t}_{1,2}}\,\mathfrak{t}_{2,3}\,\overbrace{\mathfrak{s}_{3,\alpha,f(X_1)}}\,\mathfrak{t}_{2,3}\mathfrak{t}_{1,2}$$

Using the relation $\mathfrak{t}_{1,2}\mathfrak{s}_{3,\beta,g(X_1,X_2)}\mathfrak{t}_{1,2} = \mathfrak{s}_{3,\beta,g(X_2,X_1)}$ from $\mathfrak{H}_2$:

$$= \mathfrak{t}_{1,2}\mathfrak{t}_{2,3}\mathfrak{s}_{3,\alpha,f(X_1)}{}^{-1}\,\underbrace{\mathfrak{t}_{2,3}\,\overbrace{\mathfrak{s}_{3,\beta,g(X_2,X_1)}}\,\mathfrak{t}_{2,3}}\,\mathfrak{s}_{3,\alpha,f(X_1)}\mathfrak{t}_{2,3}\mathfrak{t}_{1,2}$$

Using the relation $\mathfrak{t}_{2,3}\mathfrak{s}_{3,\beta,g(X_2,X_1)}\mathfrak{t}_{2,3} = \mathfrak{s}_{2,\beta,g(X_3,X_1)}$ from $\widetilde{\mathfrak{H}}_1$:

$$= \mathfrak{t}_{1,2}\mathfrak{t}_{2,3}\underbrace{{\mathfrak{s}_{3,\alpha,f(X_1)}}^{-1}\overbrace{\mathfrak{s}_{2,\beta,g(X_3,X_1)}}\mathfrak{s}_{3,\alpha,f(X_1)}}\mathfrak{t}_{2,3}\mathfrak{t}_{1,2}$$

Applying (19):

$$= \mathfrak{t}_{1,2}\underbrace{\mathfrak{t}_{2,3}\overbrace{\mathfrak{s}_{2,\beta,g(\alpha X_3+f(X_1),X_1)}}\mathfrak{t}_{2,3}}\mathfrak{t}_{1,2}$$

Using the relation $\mathfrak{t}_{2,3}\mathfrak{s}_{2,\beta,g(\alpha X_3+f(X_1),X_1)}\mathfrak{t}_{2,3} = \mathfrak{s}_{3,\beta,g(\alpha X_2+f(X_1),X_1)}$ from $\widetilde{\mathfrak{H}}_1$:

$$\begin{aligned}
&= \underbrace{\mathfrak{t}_{1,2}\overbrace{\mathfrak{s}_{3,\beta,g(\alpha X_2+f(X_1),X_1)}}\mathfrak{t}_{1,2}} \\
&= \mathfrak{s}_{3,\beta,g(\alpha X_1+f(X_2),X_2)} \qquad \text{from } \mathfrak{H}_2 \\
&= \widehat{\Psi}\big([\sigma_{3,\beta,g(\alpha X_1+f(X_2),X_2)}]\big) \\
&= \widehat{\Psi}\big([\sigma_{3,\beta,g(\sigma_{1,\alpha,f(X_2)})}]\big),
\end{aligned}$$

which accomplishes our goal.

Now let $i = 3$, $j = 1$. For $f \in K[X_2]$ and $g \in K[X_2, X_3]$,

$$\begin{aligned}
&\widehat{\Psi}\big([\sigma_{3,\alpha,f(X_2)}]^{-1}[\sigma_{1,\beta,g(X_2,X_3)}][\sigma_{3,\alpha,f(X_2)}]\big) \\
&\quad = \widehat{\Psi}\big([\sigma_{3,\alpha,f(X_2)}]\big)^{-1}\widehat{\Psi}\big([\sigma_{1,\beta,g(X_2,X_3)}]\big)\widehat{\Psi}\big([\sigma_{3,\alpha,f(X_2)}]\big) \\
&\quad = {\mathfrak{s}_{3,\alpha,f(X_2)}}^{-1}\mathfrak{t}_{1,3}\mathfrak{s}_{3,\beta,g(X_2,X_1)}\mathfrak{t}_{1,3}\mathfrak{s}_{3,\alpha,f(X_2)} \qquad \text{by (14) and (17)} \\
&\quad = \mathfrak{t}_{1,3}\underbrace{\mathfrak{t}_{1,3}}{\mathfrak{s}_{3,\alpha,f(X_2)}}^{-1}\underbrace{\mathfrak{t}_{1,3}}\mathfrak{s}_{3,\beta,g(X_2,X_1)}\underbrace{\mathfrak{t}_{1,3}}\mathfrak{s}_{3,\alpha,f(X_2)}\underbrace{\mathfrak{t}_{1,3}}\mathfrak{t}_{1,3} \quad \text{since } {\mathfrak{t}_{1,3}}^2 = 1
\end{aligned}$$

Applying (20):

$$\begin{aligned}
= \mathfrak{t}_{1,3}&\overbrace{\mathfrak{t}_{1,2}\mathfrak{t}_{2,3}\mathfrak{t}_{1,2}}{\mathfrak{s}_{3,\alpha,f(X_2)}}^{-1}\overbrace{\mathfrak{t}_{1,2}\mathfrak{t}_{2,3}\mathfrak{t}_{1,2}}\mathfrak{s}_{3,\beta,g(X_2,X_1)} \\
&\overbrace{\mathfrak{t}_{1,2}\mathfrak{t}_{2,3}\mathfrak{t}_{1,2}}\mathfrak{s}_{3,\alpha,f(X_2)}\overbrace{\mathfrak{t}_{1,2}\mathfrak{t}_{2,3}\mathfrak{t}_{1,2}}\mathfrak{t}_{1,3}
\end{aligned}$$

Using the relation $\mathfrak{t}_{1,2}\mathfrak{s}_{3,\alpha,f(X_2)}\mathfrak{t}_{1,2} = \mathfrak{s}_{3,\alpha,f(X_1)}$ in $\mathfrak{H}_2$:

$$= \mathfrak{t}_{1,3}\mathfrak{t}_{1,2}\underbrace{\mathfrak{t}_{2,3}\overbrace{\mathfrak{s}_{3,\alpha,f(X_1)}}^{-1}\mathfrak{t}_{2,3}}\mathfrak{t}_{1,2}\mathfrak{s}_{3,\beta,g(X_2,X_1)}\mathfrak{t}_{1,2}\underbrace{\mathfrak{t}_{2,3}\overbrace{\mathfrak{s}_{3,\alpha,f(X_1)}}\mathfrak{t}_{2,3}}\mathfrak{t}_{1,2}\mathfrak{t}_{1,3}$$

Using the relation $\mathfrak{t}_{2,3}\mathfrak{s}_{3,\alpha,f(X_1)}\mathfrak{t}_{2,3} = \mathfrak{s}_{2,\alpha,f(X_1)}$ in $\widetilde{\mathfrak{H}}_1$:

$$= \mathfrak{t}_{1,3}\mathfrak{t}_{1,2}\overbrace{\mathfrak{s}_{2,\alpha,f(X_1)}}^{-1}\underbrace{\mathfrak{t}_{1,2}\mathfrak{s}_{3,\beta,g(X_2,X_1)}\mathfrak{t}_{1,2}}\overbrace{\mathfrak{s}_{2,\alpha,f(X_1)}}\mathfrak{t}_{1,2}\mathfrak{t}_{1,3}$$

Using the relation $\mathfrak{t}_{1,2}\mathfrak{s}_{3,\beta,g(X_2,X_1)}\mathfrak{t}_{1,2} = \mathfrak{s}_{3,\beta,g(X_1,X_2)}$ in $\mathfrak{H}_2$:

$$= \mathfrak{t}_{1,3}\mathfrak{t}_{1,2}\,\underbrace{\mathfrak{s}_{2,\alpha,f(X_1)}{}^{-1}\,\overbrace{\mathfrak{s}_{3,\beta,g(X_1,X_2)}}\,\mathfrak{s}_{2,\alpha,f(X_1)}}\,\mathfrak{t}_{1,2}\mathfrak{t}_{1,3}$$

Applying (19):

$$= \mathfrak{t}_{1,3}\,\underbrace{\mathfrak{t}_{1,2}\,\overbrace{\mathfrak{s}_{3,\beta,g(X_1,\alpha X_2+f(X_1))}}\,\mathfrak{t}_{1,2}}\,\mathfrak{t}_{1,3}$$

Using the relation $\mathfrak{t}_{1,2}\mathfrak{s}_{3,\beta,g(X_1,\alpha X_2+f(X_1))}\mathfrak{t}_{1,2} = \mathfrak{s}_{3,\beta,g(X_2,\alpha X_1+f(X_2))}$ in $\mathfrak{H}_2$:

$$\begin{aligned}
&= \mathfrak{t}_{1,3}\,\overbrace{\mathfrak{s}_{3,\beta,g(X_2,\alpha X_1+f(X_2))}}\,\mathfrak{t}_{1,3}\\
&= \widehat{\Psi}\big([\sigma_{1,\beta,g(X_2,\alpha X_1+f(X_2))}]\big)\\
&= \widehat{\Psi}\big([\sigma_{1,\beta,g(\sigma_{3,\alpha,f(X_2)})}]\big) \quad \text{by (17),}
\end{aligned}$$

as desired.

The two cases $\{i,j\} = \{1,2\}$ will employ the equality

$$\mathfrak{t}_{1,3}\mathfrak{s}_{2,\beta,g(X_1,X_3)}\mathfrak{t}_{1,3} = \mathfrak{s}_{2,\beta,g(X_3,X_1)}, \tag{21}$$

which arises by conjugating the $\mathfrak{H}_2$ identity $\mathfrak{t}_{1,2}\mathfrak{s}_{3,\beta,g(X_1,X_2)}\mathfrak{t}_{1,2} = \mathfrak{s}_{3,\beta,g(X_2,X_1)}$ by $\mathfrak{t}_{2,3}$, evoking the $\widetilde{\mathfrak{H}}_1$ identity $\mathfrak{t}_{2,3}\mathfrak{s}_{3,\beta,g(X_1,X_2)}\mathfrak{t}_{2,3} = \mathfrak{s}_{2,\beta,g(X_1,X_3)}$ and the $\mathfrak{H}_3$ identity $\mathfrak{t}_{2,3}\mathfrak{t}_{1,2}\mathfrak{t}_{2,3} = \mathfrak{t}_{1,3}$.

For $i = 1, j = 2$ we have

$$\begin{aligned}
&\widehat{\Psi}\big([\sigma_{1,\alpha,f(X_3)}]^{-1}[\sigma_{2,\beta,g(X_1,X_3)}][\sigma_{1,\alpha,f(X_3)}]\big)\\
&\quad= \widehat{\Psi}\big([\sigma_{1,\alpha,f(X_3)}]\big)^{-1}\widehat{\Psi}\big([\sigma_{2,\beta,g(X_1,X_3)}]\big)\widehat{\Psi}\big([\sigma_{1,\alpha,f(X_3)}]\big)\\
&\quad= \mathfrak{t}_{1,3}\mathfrak{s}_{3,\alpha,f(X_1)}{}^{-1}\,\underbrace{\mathfrak{t}_{1,3}\mathfrak{s}_{2,\beta,g(X_1,X_3)}\mathfrak{t}_{1,3}}\,\mathfrak{s}_{3,\alpha,f(X_1)}\mathfrak{t}_{1,3} \qquad \text{by (17)}\\
&\quad= \mathfrak{t}_{1,3}\,\underbrace{\mathfrak{s}_{3,\alpha,f(X_1)}{}^{-1}\,\overbrace{\mathfrak{s}_{2,\beta,g(X_3,X_1)}}\,\mathfrak{s}_{3,\alpha,f(X_1)}}\,\mathfrak{t}_{1,3} \qquad \text{by (21)}\\
&\quad= \underbrace{\mathfrak{t}_{1,3}\,\overbrace{\mathfrak{s}_{2,\beta,g(\alpha X_3+f(X_1),X_1)}}\,\mathfrak{t}_{1,3}} \qquad \text{by (19)}\\
&\quad= \mathfrak{s}_{2,\beta,g(\alpha X_1+f(X_3),X_3)} \qquad \text{by (21)}\\
&\quad= \mathfrak{s}_{2,\beta,g(\sigma_{1,\alpha,f(X_3)})}\\
&\quad= \widehat{\Psi}\big([\sigma_{2,\beta,g(\sigma_{1,\alpha,f(X_3)})}]\big).
\end{aligned}$$

The case $i = 2, j = 1$ follows similarly:

$$\begin{aligned}
&\widehat{\Psi}\left([\sigma_{2,\alpha,f(X_3)}]^{-1}[\sigma_{1,\beta,g(X_2,X_3)}][\sigma_{2,\alpha,f(X_3)}]\right)\\
&\quad= \widehat{\Psi}\big([\sigma_{2,\alpha,f(X_3)}]\big)^{-1}\widehat{\Psi}\big([\sigma_{1,\beta,g(X_2,X_3)}]\big)\widehat{\Psi}\big([\sigma_{2,\alpha,f(X_3)}]\big)
\end{aligned}$$

$$
\begin{aligned}
&= {\mathfrak{s}_{2,\alpha,f(X_3)}}^{-1}\mathfrak{t}_{1,3}\mathfrak{s}_{3,\beta,g(X_2,X_1)}\mathfrak{t}_{1,3}\mathfrak{s}_{2,\alpha,f(X_3)} && \text{by (17)}\\
&= \mathfrak{t}_{1,3}\mathfrak{t}_{1,3}{\mathfrak{s}_{2,\alpha,f(X_3)}}^{-1}\underbrace{\mathfrak{t}_{1,3}\mathfrak{s}_{3,\beta,g(X_2,X_1)}\mathfrak{t}_{1,3}}\mathfrak{s}_{2,\alpha,f(X_3)}\mathfrak{t}_{1,3}\mathfrak{t}_{1,3} && \text{since } {\mathfrak{t}_{1,3}}^2 = 1\\
&= \mathfrak{t}_{1,3}\underbrace{{\mathfrak{s}_{2,\alpha,f(X_1)}}^{-1}\overbrace{\mathfrak{s}_{3,\beta,g(X_2,X_1)}}\mathfrak{s}_{2,\alpha,f(X_1)}}\mathfrak{t}_{1,3} && \text{by (21)}\\
&= \underbrace{\mathfrak{t}_{1,3}\overbrace{\mathfrak{s}_{3,\beta,g(\alpha X_2+f(X_1),X_1)}}\mathfrak{t}_{1,3}} && \text{by (19)}\\
&= \widehat{\Psi}\big([\sigma_{1,\beta,g(\alpha X_2+f(X_3),X_3)}]\big) && \text{by (17),}
\end{aligned}
$$

completing the proof that the relation (R2) is respected by $\widehat{\Psi}$.

Lastly we come to (R3). Here, recall that $k \neq \ell$ and j is the image of i under the permutation that switches k and ℓ. If $\{k,\ell,i\} = \{2,3\}$ then we also have $j \in \{2,3\}$ and we must show that $\mathfrak{t}_{k,\ell}\mathfrak{s}_{i,\alpha,f}\mathfrak{t}_{k,\ell} = \mathfrak{s}_{j,\alpha,f(\tau_{k,\ell})}$. But this relation holds in $\widetilde{\mathfrak{H}}_1$. Also, if $i = 3$ and $\{k,\ell\} = \{1,2\}$, then $j = 3$ and the relation holds in $\mathfrak{H}_2$. Thus for $i = 3$ the only remaining case is $\{k,\ell\} = \{1,3\}$, which follows quickly from (17). To wit:

$$
\begin{aligned}
&\widehat{\Psi}([\tau_{1,3}][\sigma_{3,\alpha,f(X_1,X_2)}][\tau_{1,3}])\\
&\qquad = \widehat{\Psi}([\tau_{1,3}])\widehat{\Psi}([\sigma_{3,\alpha,f(X_1,X_2)}])\widehat{\Psi}([\tau_{1,3}])\\
&\qquad = \mathfrak{t}_{1,3}\mathfrak{s}_{3,\alpha,f(X_1,X_2)}\mathfrak{t}_{1,3} && \text{by (14) and (16)}\\
&\qquad = \widehat{\Psi}\big([\sigma_{1,\alpha,f(X_3,X_2)}]\big) && \text{by (17)}\\
&\qquad = \widehat{\Psi}\big([\sigma_{1,\alpha,f(\tau_{1,3})}]\big).
\end{aligned}
$$

For $i = 2$ the remaining cases are $\{k,\ell\} = \{1,2\}$ and $\{k,\ell\} = \{1,3\}$. For the first:

$$
\begin{aligned}
&\widehat{\Psi}([\tau_{1,2}][\sigma_{2,\alpha,f(X_1,X_3)}][\tau_{1,2}])\\
&\qquad = \widehat{\Psi}([\tau_{1,2}])\widehat{\Psi}([\sigma_{2,\alpha,f(X_1,X_3)}])\widehat{\Psi}([\tau_{1,2}])\\
&\qquad = \mathfrak{t}_{1,2}\underbrace{\mathfrak{s}_{2,\alpha,f(X_1,X_3)}}\mathfrak{t}_{1,2} && \text{by (14) and (16)}
\end{aligned}
$$

Using the relation $\mathfrak{s}_{2,\alpha,f(X_1,X_3)} = \mathfrak{t}_{2,3}\mathfrak{s}_{3,\alpha,f(X_1,X_2)}\mathfrak{t}_{2,3}$ in $\widetilde{\mathfrak{H}}_1$:

$$
\begin{aligned}
&= \underbrace{\mathfrak{t}_{1,2}\mathfrak{t}_{2,3}}\mathfrak{s}_{3,\alpha,f(X_1,X_2)}\underbrace{\mathfrak{t}_{2,3}\mathfrak{t}_{1,2}}\\
&= \overbrace{\mathfrak{t}_{1,3}\mathfrak{t}_{1,2}}\mathfrak{s}_{3,\alpha,f(X_1,X_2)}\overbrace{\mathfrak{t}_{1,2}\mathfrak{t}_{1,3}} && \text{using (20)}
\end{aligned}
$$

Using the relation $\mathfrak{t}_{1,2}\mathfrak{s}_{3,\alpha,f(X_1,X_2)}\mathfrak{t}_{1,2} = \mathfrak{s}_{3,\alpha,f(X_2,X_1)}$ in $\mathfrak{H}_2$:

$$
\begin{aligned}
&= \mathfrak{t}_{1,3}\overbrace{\mathfrak{s}_{3,\alpha,f(X_2,X_1)}}\mathfrak{t}_{1,3}\\
&= \widehat{\Psi}\left([\sigma_{1,\alpha,f(X_2,X_3)}]\right) && \text{by (17)}\\
&= \widehat{\Psi}\left([\sigma_{1,\alpha,f(\tau_{1,2})}]\right).
\end{aligned}
$$

For the second:

$$
\begin{aligned}
&\widehat{\Psi}([\tau_{1,3}][\sigma_{2,\alpha,f(X_1,X_3)}][\tau_{1,3}]) \\
&\qquad = \widehat{\Psi}([\tau_{1,3}])\widehat{\Psi}([\sigma_{2,\alpha,f(X_1,X_3)}])\widehat{\Psi}([\tau_{1,3}]) \\
&\qquad = \mathfrak{t}_{1,3}\mathfrak{s}_{2,\alpha,f(X_1,X_3)}\mathfrak{t}_{1,3} \qquad \text{by (14) and (16)} \\
&\qquad = \mathfrak{s}_{2,\alpha,f(X_3,X_1)} \qquad \text{by (21)} \\
&\qquad = \widehat{\Psi}\big([\sigma_{2,\alpha,f(X_3,X_1)}]\big) \\
&\qquad = \widehat{\Psi}\big([\sigma_{2,\alpha,f(\tau_{1,3})}]\big).
\end{aligned}
$$

Thus we have verified all the cases when $i = 2$ or $i = 3$.

Finally we consider $i = 1$. If $\{k,\ell\} = \{2,3\}$, (R3) is a consequence of (17):

$$
\begin{aligned}
&\widehat{\Psi}([\tau_{2,3}][\sigma_{1,\alpha,f(X_2,X_3)}][\tau_{2,3}]) \\
&\qquad = \widehat{\Psi}([\tau_{2,3}])\widehat{\Psi}([\sigma_{1,\alpha,f(X_2,X_3)}])\widehat{\Psi}([\tau_{2,3}]) \\
&\qquad = \underbrace{\mathfrak{t}_{2,3}\mathfrak{t}_{1,3}}\,\mathfrak{s}_{3,\alpha,f(X_2,X_1)}\,\underbrace{\mathfrak{t}_{1,3}\mathfrak{t}_{2,3}} \qquad \text{by (16) and (17)} \\
&\qquad = \overbrace{\mathfrak{t}_{1,3}\mathfrak{t}_{1,2}}\mathfrak{s}_{3,\alpha,f(X_2,X_1)}\,\overbrace{\mathfrak{t}_{1,2}\mathfrak{t}_{1,3}} \qquad \text{using (20)}
\end{aligned}
$$

Using the relation $\mathfrak{t}_{1,2}\mathfrak{s}_{3,\alpha,f(X_2,X_1)}\mathfrak{t}_{1,2} = \mathfrak{s}_{3,\alpha,f(X_1,X_2)}$ in $\mathfrak{H}_2$:

$$
\begin{aligned}
&\qquad = \mathfrak{t}_{1,3}\,\overbrace{\mathfrak{s}_{3,\alpha,f(X_1,X_2)}}\,\mathfrak{t}_{1,3} \\
&\qquad = \widehat{\Psi}\big([\sigma_{1,\alpha,f(X_3,X_2)}]\big) \qquad \text{by (17)} \\
&\qquad = \widehat{\Psi}\big([\sigma_{1,\alpha,f(\tau_{2,3})}]\big).
\end{aligned}
$$

If $\{k,\ell\} = \{1,3\}$ we have

$$
\begin{aligned}
&\widehat{\Psi}([\tau_{1,3}][\sigma_{1,\alpha,f(X_2,X_3)}][\tau_{1,3}]) \\
&\qquad = \widehat{\Psi}([\tau_{1,3}])\widehat{\Psi}([\sigma_{1,\alpha,f(X_2,X_3)}])\widehat{\Psi}([\tau_{1,3}]) \\
&\qquad = \mathfrak{t}_{1,3}\mathfrak{t}_{1,3}\mathfrak{s}_{3,\alpha,f(X_2,X_1)}\mathfrak{t}_{1,3}\mathfrak{t}_{1,3} \qquad \text{by (16) and (17)} \\
&\qquad = \mathfrak{s}_{3,\alpha,f(X_2,X_1)} \qquad \text{since } {\mathfrak{t}_{1,3}}^2 = 1 \\
&\qquad = \widehat{\Psi}\big([\sigma_{3,\alpha,f(X_2,X_1)}]\big) \qquad \text{by (17)} \\
&\qquad = \widehat{\Psi}\big([\sigma_{3,\alpha,f(\tau_{1,3})}]\big).
\end{aligned}
$$

If $\{k,\ell\} = \{1,2\}$ we have

$$
\begin{aligned}
&\widehat{\Psi}([\tau_{1,2}][\sigma_{1,\alpha,f(X_2,X_3)}][\tau_{1,2}]) \\
&\qquad = \widehat{\Psi}([\tau_{1,2}])\widehat{\Psi}([\sigma_{1,\alpha,f(X_2,X_3)}])\widehat{\Psi}([\tau_{1,2}]) \\
&\qquad = \underbrace{\mathfrak{t}_{1,2}\mathfrak{t}_{1,3}}\,\mathfrak{s}_{3,\alpha,f(X_2,X_1)}\,\underbrace{\mathfrak{t}_{1,3}\mathfrak{t}_{1,2}} \qquad \text{by (16) and (17)}
\end{aligned}
$$

$$= \overbrace{\mathfrak{t}_{1,3}\mathfrak{t}_{2,3}}\mathfrak{s}_{3,\alpha,f(X_2,X_1)}\overbrace{\mathfrak{t}_{2,3}\mathfrak{t}_{1,3}} \qquad \text{using (20)}$$

Using the relation $\mathfrak{t}_{2,3}\mathfrak{s}_{3,\alpha,f(X_2,X_1)}\mathfrak{t}_{2,3} = \mathfrak{s}_{2,\alpha,f(X_3,X_1)}$ in $\widetilde{\mathfrak{H}}_1$:

$$\begin{aligned}
&= \mathfrak{t}_{1,3}\,\overbrace{\mathfrak{s}_{2,\alpha,f(X_3,X_1)}}\,\mathfrak{t}_{1,3} \\
&= \mathfrak{s}_{2,\alpha,f(X_1,X_3)} \qquad \text{by (21)} \\
&= \widehat{\Psi}\big([\sigma_{2,\alpha,f(X_1,X_3)}]\big) \\
&= \widehat{\Psi}\big([\sigma_{2,\alpha,f(\tau_{1,2})}]\big).
\end{aligned}$$

We have now shown that $\mathcal{N}$ lies in the kernel of $\widehat{\Psi}$, and therefore there is an induced map $\Psi : \mathrm{TA}_3(K) \to \mathfrak{G}$ identifying via the isomorphism of (13), which we will now show is inverse to the map $\Phi : \mathfrak{G} \to \mathrm{TA}_3(K)$. This follows from the following easy fact: Each of the groups $\widetilde{H}_1, H_2$, and H_3 are generated by the elements $\sigma_{i,a,f}$ that lie within it. For H_2 and H_3 this is straightforward, and for $\widetilde{H}_1$ it follows from (11). Therefore the elements $\sigma_{i,a,f}$ that lie in $\widetilde{H}_1 \cup H_2 \cup H_3$ generate $\mathrm{TA}_3(K)$, and the corresponding elements $\mathfrak{s}_{i,a,f}$ in $\widetilde{\mathfrak{H}}_1 \cup \mathfrak{H}_2 \cup \mathfrak{H}_3$ generate $\mathfrak{H}$ as well. These are the elements for which $i \in \{2,3\}$, or $i = 1$ and $\deg f \leq 1$. The assignments (14) and (15) show that $\Psi(\sigma_{i,a,f}) = \mathfrak{s}_{i,a,f}$ in these cases, and it is clear that $\Phi(\mathfrak{s}_{i,a,f}) = \sigma_{i,a,f}$ by the definition of Φ. This shows that Ψ and Φ are inverses, completing the proof of Theorem 2.

6. Concluding remarks and questions

The combinatoric upshot of Theorem 2 is that $\mathrm{TA}_3(K)$ is the colimit of a "triangle of groups" in Stallings' sense (see [12]), comprising $\widetilde{H}_1$, H_2, and H_3, their pairwise intersections, and the intersection of all three. These groups form the stabilizers of the three vertices, three edges, and face, respectively, of a simplex $\mathfrak{f}$ in a 2-dimensional simply connected simplicial complex $\mathcal{D}$ on which $\mathrm{TA}_3(K)$ acts, and for which $\mathfrak{f}$ serves as a fundamental domain. There are unanswered questions about $\mathcal{D}$. For example, is it 2-connected (i.e., is $\pi_2(\mathcal{D}) = 0$), and does it have infinite diameter (i.e., does the 1-skeleton of $\mathcal{D}$ have infinite diameter as a graph)?

The 3-dimensional simplicial complex $\mathcal{D}$ can be realized as follows: More generally we construct an n-dimensional simplicial complex $\mathcal{E}_n$ whose vertices are rank $(i+1)$ vector spaces V in $K[X_1, \ldots, X_n]$ containing K, where $1 \leq i \leq n$ such that $K[X_1, \ldots, X_n] = K[V]^{[n-i]}$. (Here $K[V]$ is the subalgebra generated by V.) Vertices $V_1, \ldots, V_r$ of strictly ascending rank form an r-simplex if $V_1 \subset \cdots \subset V_r$. There is an obvious action of $\mathrm{GA}_n(K)$ on $\mathcal{E}_n$. Note that $\mathcal{E}_n$ contains the n-simplex Σ_n determined by the n vertices V_i, $1 \leq i \leq n$, defined by (6) in Section 4. This is a fundamental domain for the action, and the subgroup H_i is the stabilizer of V_i, by its definition (7). For $n = 2$ this is the tree which gives the structure Theorem for $\mathrm{GA}_2(K)$ (Example 4).

For $n \geq 3$ we do not know if $\mathcal{E}_n$ is connected, or simply connected. The connectivity of $\mathcal{E}_n$ is equivalent to the generation of $\mathrm{GA}_n(K)$ by the subgroups $H_1, \ldots, H_n$, an unsolved question.

By restriction, $\mathrm{TA}_n(K)$ also acts on $\mathcal{E}_n$. The stabilizer of V_i in $\mathrm{TA}_n(K)$ is $\widetilde{H}_i$, by (9). Let $\mathcal{D}_n$ be the subcomplex consisting of the $\mathrm{TA}_n(K)$-translates of the simplex Σ_n. (For $n = 3$, this is the complex $\mathcal{D}$ mentioned above.) The fact that $\mathrm{TA}_n(K)$ is generated by the stabilizers $\widetilde{H}_i$ implies $\mathcal{D}_n$ is connected. Simple connectivity of $\mathcal{D}_n$ is equivalent to the assertion that $\mathrm{TA}_n(K)$ is the generalized amalgamated product of $\widetilde{H}_, \dots, \widetilde{H}_n$ along pairwise intersections. For $n = 3$ this is true by the main result of this paper; for $n \geq 4$ it is unknown.

References

[1] J. Berson, A. van den Essen, D. Wright, *Stable tameness of two-dimensional polynomial automorphisms over a regular ring*, Adv. Math. **230** (2012), no. 4–6, 2176–2197.

[2] J. M. Corson, *Complexes of groups*, Proc. London Math. Soc. (3) **65** (1992), no. 1, 199–224.

[3] G. Freudenburg. P. Russell, *Open problems in affine algebraic geometry*, in: *Affine Algebraic Geometry*, Contemp. Math., Vol. 369, Amer. Math. Soc., Providence, RI, 2005, pp. 1–30.

[4] G. Freudenburg, *Actions of* $\mathbf{G}_a$ *on* $\mathbf{A}^3$ *defined by homogeneous derivations*, J. Pure Appl. Algebra **126** (1998), no. 1–3, 169–181.

[5] A. Haefliger, *Complexes of groups and orbihedra*, in: *Group Theory from a Geometrical Viewpoint* (Trieste, 1990), World Sci. Publ., River Edge, NJ, 1991, pp. 504–540.

[6] H W. E. Jung, *Über ganze birationale Transformationen der Ebene*, J. Reine Angew. Math. **184** (1942), 161–174.

[7] W. van der Kulk,*On polynomial rings in two variables*, Nieuw Arch. Wiskunde (3) **1** (1953), 33–41.

[8] W. Magnus, A. Karrass, D. Solitar, *Combinatorial Group Theory. Presentations of Groups in Terms of Generators and Relations*, 2nd ed., Dover, Mineola, NY, 2004.

[9] H. Nagao, *On* $\mathrm{GL}(2, K[x])$, J. Inst. Polytech. Osaka City Univ. Ser. A **10** (1959), 117–121.

[10] J.-P. Serre, *Trees*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003.

[11] I. P. Shestakov, U. U. Umirbaev, *The tame and the wild automorphisms of polynomial rings in three variables*, J. Amer. Math. Soc. **17** (2004), no. 1, 197–227.

[12] J. R. Stallings, *Non-positively curved triangles of groups*, in: *Group Theory from a Geometrical Viewpoint* (Trieste, 1990), World Sci. Publ., River Edge, NJ, 1991, pp. 491–503.

[13] У. У. Умирбаев, *Определяющие соотношения группы ручных автоморфизмов колец многочленов и дикие автоморфизмы свободных ассоциативных алгебр*, Докл. Акад. Наук **407** (2006), no. 3, 319–324. [U. U. Umirbaev, *Defining relations of the tame automorphism group of polynomial rings, and wild automorphisms of free associative algebras*, Dokl. Akad. Nauk **407** (2006), no. 3, 319–324 (Russian).]

[14] D. Wright, *Two-dimensional Cremona groups acting on simplicial complexes*, Trans. Amer. Math. Soc. **331** (1992), no. 1, 281–300.