The Jordan Constant For Cremona Group of Rank 2

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ABSTRACT. We compute the Jordan constant for the group of birational automorphisms of a projective plane $\mathbb{P}^2_{\Bbbk}$, where $\Bbbk$ is either an algebraically closed field of characteristic 0, or the field of real numbers, or the field of rational numbers.

1. INTRODUCTION

1.1. **Jordan property.** Throughout the paper $\Bbbk$ denotes an algebraically closed field of characteristic zero, unless stated otherwise. We start with the main definition of this article.

Definition 1.1. A group Γ is called *Jordan* (we also say that Γ has *Jordan property*) if there exists a positive integer m such that every finite subgroup $G \subset \Gamma$ contains a normal abelian subgroup $A \triangleleft G$ of index at most m. The minimal such m is called the *Jordan constant* of Γ and is denoted by $\mathrm{J}(\Gamma)$.

Informally, this means that all finite subgroups of Γ are "almost" abelian. The name of $\mathrm{J}(\Gamma)$ (and the corresponding property) is justified by the classical theorem of Camille Jordan.

Theorem 1.2 (C. Jordan, 1878)**.** *The group* $\mathrm{GL}_n(\Bbbk)$ *is Jordan for every* n.

Since any subgroup of a Jordan group is obviously Jordan, Theorem 1.2 implies that every linear algebraic group over $\Bbbk$ is Jordan. In recent years the Jordan property has been studied for groups of birational automorphisms of algebraic varieties. The first significant result in this direction belongs to J.-P. Serre. Before stating it, let us recall that the *Cremona group* $\mathrm{Cr}_n(\Bbbk)$ of rank n is the group of birational automorphisms of a projective space $\mathbb{P}^n_{\Bbbk}$ (or, equivalently, the group of $\Bbbk$-automorphisms of the field $\Bbbk(x_1,\ldots,x_n)$ of rational functions in n independent variables). Note that $\mathrm{Cr}_1(\Bbbk) \cong \mathrm{PGL}_2(\Bbbk)$ is linear and hence is Jordan, but the group $\mathrm{Cr}_2(\Bbbk)$ is already "very far" from being linear. However, the following holds.

Theorem 1.3 (J.-P. Serre [Ser09, Theorem 5.3],[Ser08, Théorème 3.1])**.** *The Cremona group* $\mathrm{Cr}_2(\Bbbk)$ *over a field* $\Bbbk$ *of characteristic 0 is Jordan.*

A far-going generalization of Theorem 1.3 was recently proved by Yu. Prokhorov and C. Shramov. To state their result in its full generality, we first need to recall the following statement, also known as Borisov–Alexeev–Borisov conjecture.

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Conjecture 1.4. *For a given positive integer n, Fano varieties of dimension n with terminal singularities are bounded, i. e. are contained in a finite number of algebraic families.*

Modulo this conjecture, we have the following strong result.

Theorem 1.5. [PS16a, Theorem 1.8] *Assume that Conjecture 1.4 holds in dimension n. Then there is a constant $I = I(n)$ such that for any rationally connected variety X of dimension n defined over an arbitrary field $\Bbbk$ of characteristic 0 and for any finite subgroup $G \subset \mathrm{Bir}(X)$ there exists a normal abelian subgroup $A \subset G$ of index at most I.*

Note that this theorem states not only that $\mathrm{Bir}(X)$ are Jordan, but also that the corresponding constant may be chosen uniformly for all rationally connected X of a fixed dimension. Conjecture 1.4 is settled in dimensions $\leqslant 3$, so the space Cremona group $\mathrm{Cr}_3(\Bbbk)$ is known to be Jordan. At this writing (October 2016), the Borisov-Alexeev-Borisov conjecture seems to be proved in all dimensions in a recent preprint of Caucher Birkar [Bir16]. So, one has

Corollary 1.6. *The group $\mathrm{Cr}_n(\Bbbk)$ is Jordan for each $n \geqslant 1$.*

1.2. **Jordan constant.** So far we discussed only the Jordan property itself. Of course, after establishing that a given group is Jordan, the next natural question is to estimate its Jordan constant. This can be highly non-trivial: the precise values of $\mathrm{J}(\mathrm{GL}_n(\Bbbk))$ for all n were found only in 2007 by M. J. Collins [Col07]. As for Cremona groups, in [Ser09] a *"multiplicative"* upper bound for $\mathrm{J}(\mathrm{Cr}_2(\Bbbk))$ is given. Specifically, it was shown that every finite subgroup $G \subset \mathrm{Cr}_2(\Bbbk)$ contains a normal abelian subgroup A with $[G : A]$ dividing $2^{10} \cdot 3^4 \cdot 5^2 \cdot 7$.

For any group Γ one can consider a closely related constant $\bar{\mathrm{J}}(\Gamma)$, which is called a *weak Jordan constant* in [PS16b]. By definition, it is the minimal number m such that for any finite subgroup $G \subset \Gamma$ there exists a *not necessarily normal* abelian subgroup $A \subset G$ of index at most m. One can show that

$$\bar{\mathrm{J}}(\Gamma) \leqslant \mathrm{J}(\Gamma) \leqslant \bar{\mathrm{J}}(\Gamma)^2$$

for any Jordan group Γ. The weak Jordan constants for $\mathrm{Cr}_2(\Bbbk)$ and $\mathrm{Cr}_3(\Bbbk)$ were computed by Prokhorov and Shramov.

Theorem 1.7. [PS16b, Proposition 1.2.3, Theorem 1.2.4] *Suppose that the field $\Bbbk$ has characteristic 0. Then one has*

$$\bar{\mathrm{J}}(\mathrm{Cr}_2(\Bbbk)) \leqslant 288, \qquad \bar{\mathrm{J}}(\mathrm{Cr}_3(\Bbbk)) \leqslant 10\,368.$$

These bounds become equalities if the base field $\Bbbk$ is algebraically closed.

As we noticed above, one can also obtain an upper bound for $\mathrm{J}(\mathrm{Cr}_2(\Bbbk))$ from this theorem, namely

$$\mathrm{J}(\mathrm{Cr}_2(\Bbbk)) \leqslant 82\,944 = 288^2.$$

We will show that this bound is very far from being sharp. The goal of this paper is to compute an exact Jordan constant for the plane Cremona group. Our first main theorem is

Theorem 1.8. *Let $\Bbbk$ be an algebraically closed field of characteristic 0. Then*

$$\mathrm{J}(\mathrm{Cr}_2(\Bbbk)) = 7200.$$

Further, we compute the Jordan constant for Cremona group $\mathrm{Cr}_2(\mathbb{R})$ and, as a by-product, for the group $\mathrm{Cr}_2(\mathbb{Q})$.

Theorem 1.9. *One has*

$$\mathrm{J}(\mathrm{Cr}_2(\mathbb{R})) = 120, \quad \bar{\mathrm{J}}(\mathrm{Cr}_2(\mathbb{R})) = 20.$$

Theorem 1.10. *One has*

$$\mathrm{J}(\mathrm{Cr}_2(\mathbb{Q})) = 120, \quad \bar{\mathrm{J}}(\mathrm{Cr}_2(\mathbb{Q})) = 20.$$

Let us consider the category whose objects are $\mathbb{R}$-schemes and morphisms are defined as follows: we say that there is a morphism $f : X \dashrightarrow Y$ if f is a rational map defined at all *real* points of X. Automorphisms in such a category are called *birational diffeomorphisms* and the corresponding group is denoted by $\mathrm{Aut}(X(\mathbb{R}))$. In recent years, birational diffeomorphisms of real rational projective surfaces have been studied intensively. As a by-product of Theorem 1.9, we will get the Jordan constant for the group of birational diffeomorphisms of $\mathbb{P}^2_{\mathbb{R}}$ and the sphere $\mathbb{S}^2$ viewed as the real locus $\mathrm{Q}_{3,1}(\mathbb{R})$ of the 2-dimensional quadric

$$\mathrm{Q}_{3,1} = \left\{[x_0 : x_1 : x_2 : x_3] : x_0^2 + x_1^2 + x_2^2 - x_3^2 = 0\right\} \subset \mathbb{P}^3_{\mathbb{R}}.$$

Theorem 1.11. *The following holds:*

$$\mathrm{J}\big(\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))\big) = 60, \quad \bar{\mathrm{J}}\big(\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))\big) = 12,$$
$$\mathrm{J}(\mathrm{Q}_{3,1}(\mathbb{R})) = 60, \quad \bar{\mathrm{J}}(\mathrm{Q}_{3,1}(\mathbb{R})) = 12.$$

Notation. Our notation is mostly standard.

- $\mathfrak{S}_n$ denotes the symmetic group of degree n;
- $\mathfrak{A}_n$ denotes the alternating group of degree n;
- D_n denotes the dihedral group of order $2n$;
- C_n, $n > 2$, denotes a cyclic characteristic subgroup $\mathrm{C}_n \subset \mathrm{D}_n$ of index 2;
- For a scheme X over $\mathbb{R}$ we denote by $X_{\mathbb{C}}$ its complexification

$$X_{\mathbb{C}} = X \times_{\mathrm{Spec}\,\mathbb{R}} \mathrm{Spec}\,\mathbb{C}$$

- $\langle a \rangle$ denotes a cyclic group generated by a;
- $A_{\bullet}B$ denotes an extension of B with help of a normal subgroup A.

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2. Some auxiliary results

In this short section we collect some useful facts concerning Jordan property.

Lemma 2.1. *The following assertions hold.*

(1) If Γ_1 is a subgroup of a Jordan group Γ_2, then Γ_1 is Jordan and $\mathrm{J}(\Gamma_1) \leqslant \mathrm{J}(\Gamma_2)$.
(2) If Γ_1 is a Jordan group, and there is a surjective homomorphism $\Gamma_1 \to \Gamma_2$, then Γ_2 is also Jordan with $\mathrm{J}(\Gamma_2) \leqslant \mathrm{J}(\Gamma_1)$.

The proofs are elementary and we omit them. Next let us compute some Jordan constants.

Lemma 2.2. *One has*

(1) $\mathrm{J}(\mathrm{GL}_2(\Bbbk)) = 60$.
(2) $\mathrm{J}(\mathrm{GL}_3(\Bbbk)) = 360$.
(3) $\mathrm{J}(\mathrm{PGL}_2(\Bbbk)) = 60$.
(4) $\mathrm{J}(\mathrm{PGL}_3(\Bbbk)) = 360$.

Proof. For (1) and (2) we refer the reader to [Col07], where $\mathrm{J}(\mathrm{GL}_n(\Bbbk))$ is computed for each n. To prove (3) and (4), we apply Lemma 2.1 (2) to the natural surjections $\mathrm{GL}_n(\Bbbk) \to \mathrm{PGL}_n(\Bbbk)$, $n = 2, 3$, and get $\mathrm{J}(\mathrm{PGL}_2(\Bbbk)) \leqslant 60$ and $\mathrm{J}(\mathrm{PGL}_3(\Bbbk)) \leqslant 360$. The required equalities are given by the simple groups $\mathfrak{A}_5$ and $\mathfrak{A}_6$, respectively (alternatively, one can use the well-known classification of finite subgroups of $\mathrm{PGL}_2(\Bbbk)$ and $\mathrm{PGL}_3(\Bbbk)$, like in [PS16b, Lemma 2.3.1]). □

3. The case of algebraically closed field

In the remaining part of the paper we shall use the standard language of G-varieties (see e.g. [DI09a]). We are going to deduce our main Theorem 1.8 by regularizing the action of each finite subgroup $G \subset \mathrm{Cr}_2(\Bbbk)$ on some $\Bbbk$-rational surface X. Then, applying to X the G-Minimal Model Program, we reduce to the case when X is either a *del Pezzo surface*, or a *conic bundle*.

First, we focus on del Pezzo surfaces, which by definition are projective algebraic surfaces X with ample anticanonical class $-K_X$.

Proposition 3.1. *Let X be a smooth del Pezzo surface. Then*

$$\mathrm{J}(\mathrm{Aut}(X)) \leqslant 7200.$$

If X is not isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, then one has $\mathrm{J}(\mathrm{Aut}(X)) \leqslant 360$.

Proof. We shall consider each $d = K_X^2$ separately.

$d = 9$: Then $X \cong \mathbb{P}^2$ and $\mathrm{J}(\mathrm{Aut}(X)) \leqslant 360$ by Lemma 2.2 (4).

$d = 8$: If X is a blow up $\pi : X \to \mathbb{P}^2$ at one point, then π is $\mathrm{Aut}(X)$-equivariant, so $\mathrm{J}(\mathrm{Aut}(X)) \leqslant 360$ by the case $d = 9$. Now let $X \cong \mathbb{P}^1 \times \mathbb{P}^1$. Then

$$\mathrm{Aut}(X) \cong \big(\mathrm{PGL}_2(\Bbbk) \times \mathrm{PGL}_2(\Bbbk)\big) \rtimes \mathbb{Z}/2.$$

The constant 7200 in the statement of Proposition is achieved for the group $G = (\mathfrak{A}_5 \times \mathfrak{A}_5) \rtimes \mathbb{Z}/2$, which has no normal abelian subgroups. Indeed, if A is normal in G, then $A \cap (\mathfrak{A}_5 \times \mathfrak{A}_5)$ is normal in $\mathfrak{A}_5 \times \mathfrak{A}_5$. But every normal subgroup in the direct product of two simple non-abelian groups H and K is one of the groups $1_H \times 1_K$, $1_H \times K$, $H \times 1_K$, $H \times K$. If A is abelian, it must be trivial.

$d = 7$**:** Then X is a blow up $\pi : X \to \mathbb{P}^2$ at two points, and π is again $\mathrm{Aut}(X)$-equivariant. Therefore, $\mathrm{J}(\mathrm{Aut}(X)) \leqslant 360$.

$d = 6$**:** Then X is isomorphic to the surface obtained by blowing up $\mathbb{P}^2_{\Bbbk}$ in three noncollinear points p_1, p_2, p_3. The set of (-1)-curves on X consists of six curves which form a hexagon Σ of lines in the anticanonical embedding $X \hookrightarrow \mathbb{P}^6_{\Bbbk}$. One can easily show that $\mathrm{Aut}(X) \cong (\Bbbk^*)^2 \rtimes \mathrm{D}_6$, where the torus $(\Bbbk^*)^2$ comes from automorphisms of $\mathbb{P}^2_{\Bbbk}$ that fix all the points p_i, and D_6 is the symmetry group of Σ. Therefore, $\mathrm{J}(\mathrm{Aut}(X)) \leqslant |\mathrm{D}_6| = 12$.

$d = 5$**:** Then $\mathrm{Aut}(X) \cong \mathfrak{S}_5$, so $\mathrm{J}(\mathrm{Aut}(X)) = 120$.

$d = 4$**:** Then $\mathrm{Aut}(X) \cong (\mathbb{Z}/2)^4 \rtimes \Gamma$, where $|\Gamma| \leqslant 10$ [Dol12, Theorem 8.6.8]. Thus $\mathrm{J}(\mathrm{Aut}(X)) \leqslant 10$ and we are done.

$d = 3$**:** Then either $|\mathrm{Aut}(X)| \leqslant 120$, or $\mathrm{Aut}(X) \cong (\mathbb{Z}/3)^3 \rtimes \mathfrak{S}_4$ (and X is the Fermat cubic surface) [Dol12, Theorem 9.5.8]. But in the latter case $\mathrm{J}(\mathrm{Aut}(X)) \leqslant 24$.

$d = 2$**:** Then $|\mathrm{Aut}(X)| \leqslant 336$ [Dol12, Table 8.9] and the assertion follows.

$d = 1$**:** Then $|\mathrm{Aut}(X)| \leqslant 144$ [Dol12, Table 8.14], so we are done.

□

The second type of rational surfaces we shall work with are conic bundles. Recall, that a smooth G-surface (X, G) admits a conic bundle structure, if there is a G-morphism $\pi : X \to B$, where B is a smooth curve and each scheme fibre is isomorphic to a reduced conic in $\mathbb{P}^2$.

Let us also fix some notation. In this paper every automorphism of a conic bundle $\pi : X \to B$ is supposed to preserve the conic bundle structure π. We shall write $\mathrm{Aut}(X, \pi)$ for the corresponding automorphism group. For every finite subgroup $G \subset \mathrm{Aut}(X, \pi)$ there exists a short exact sequence of groups

$$1 \longrightarrow G_F \longrightarrow G \xrightarrow{\varphi} G_B \longrightarrow 1,$$

where $G_B \subset \mathrm{Aut}(B) \cong \mathrm{PGL}_2(\Bbbk)$, and G_F acts by automorphisms of the generic fiber F. Since G is finite, G_F is a subgroup of $\mathrm{PGL}_2(\Bbbk)$. The following result will be used in the proof of Proposition 3.3. We sketch the proof for the reader's convenience.

Lemma 3.2 ([Ser09, Lemma 5.2])**.** *Let $g \in G$ and $h \in G_F$ be such that g normalizes the cyclic group $\langle h \rangle$ generated by h. Then ghg^{-1} is equal to h or to h^{-1}.*

Proof. We may assume that the order n of h is greater than 2. The automorphism h has two fixed points on F, which can be characterized by the eigenvalue of h on their tangent spaces. Denote one of these eigenvalues by λ; the other is λ^{-1} (these are primitive n-th roots of unity). The pair $\{\lambda, \lambda^{-1}\}$

is canonically associated with h, so the pair associated with ghg^{-1} is also $\{\lambda,\lambda^{-1}\}$. But $ghg^{-1}=h^k$, hence the pair associated with h^k is $\{\lambda^k,\lambda^{-k}\}$. Hence $k\equiv\pm1 \mod n$. □

Proposition 3.3. *Let X be a smooth rational surface with a conic bundle structure $\pi: X\to B\cong\mathbb{P}^1$. Then*

$$\mathrm{J}(\mathrm{Aut}(X,\pi))\leqslant 7200.$$

Proof. The argument is essentially due to J.-P. Serre, [Ser09, Theorem 5.1]. Let $G\subset\mathrm{Aut}(X,\pi)$ be a finite group. Then both G_F and G_B contain cyclic *characteristic* subgroups $G'_F=\langle h\rangle$ and G'_B of index at most 60. Pick $g\in G$ such that $\langle\varphi(g)\rangle=G'_B$. Since G'_F is normal in G, it is normalized by g. Thus, by Lemma 3.2, either $ghg^{-1}=h$, or $ghg^{-1}=h^{-1}$. In both cases g^2 commutes with h. The abelian subgroup $\langle g^2,h\rangle$ is normal in G, and from the inclusions

$$\langle g^2,h\rangle\subset\langle g,h\rangle\subset(G_F)_\bullet\langle g\rangle\subset G$$

we see that its index is at most $2\cdot60\cdot60=7200$. □

Corollary 3.4. *Let X be a smooth rational surface. Then* $\mathrm{J}(\mathrm{Aut}(X))\leqslant 7200$.

Proof. Take a finite subgroup $G\subset\mathrm{Aut}(X)$. Applying to X the G-Minimal Model Program, we may assume that X is either a del Pezzo surface, or a rational surface with G-equivariant conic bundle structure [DI09b, Theorem 5]. Now the statement follows from Propositions 3.1 and 3.3. □

Corollary 3.5 (Theorem 1.8)**.** *One has* $\mathrm{J}(\mathrm{Cr}_2(\Bbbk))=7200$.

Proof. Take a finite subgroup $G\subset\mathrm{Cr}_2(\Bbbk)$. Regularizing its action (see [DI09a, Lemma 3.5]), we may assume that G acts biregularly on a smooth rational surface X. So, the bound $\mathrm{J}(\mathrm{Cr}_2(\Bbbk))\leqslant7200$ follows from Corollary 3.4. The equality is achieved for the group $(\mathfrak{A}_5\times\mathfrak{A}_5)\rtimes\mathbb{Z}/2$ acting on $\mathbb{P}^1\times\mathbb{P}^1$. □

4. The Jordan constants for the plane Cremona groups over $\mathbb{R}$ and $\mathbb{Q}$

In recent years, growing attention has been paid to the group $\mathrm{Cr}_2(\mathbb{R})$. In contrast with $\mathrm{Cr}_2(\mathbb{C})$, there are only partial classification results for its finite subgroups at the moment (see [Yas16]). However we are still able to calculate the Jordan constant $\mathrm{J}(\mathrm{Cr}_2(\mathbb{R}))$. As a by-product result, we also get the Jordan constant for a closely related group $\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$ of birational diffeomorphisms of $\mathbb{P}^2_{\mathbb{R}}$ (see Introduction), and for the group $\mathrm{Cr}_2(\mathbb{Q})$.

Of course, from Lemma 2.1 (1) we immediately get

$$\mathrm{J}(\mathrm{Cr}_2(\mathbb{R}))\leqslant\mathrm{J}(\mathrm{Cr}_2(\mathbb{C}))=7200.$$

Using some elementary representation theory arguments (Lemma 4.1), this bound can be drastically improved. The next result is classical and we omit the proof.

Lemma 4.1. *The following assertions hold.*

(1) *Any finite subgroup of* $\mathrm{GL}_2(\mathbb{R})$ *and* $\mathrm{PGL}_2(\mathbb{R})$ *is isomorphic either to* $\mathbb{Z}/n$ *or* D_n *($n\geqslant2$).*

(2) *One has* $\mathrm{PGL}_3(\mathbb{R}) \cong \mathrm{SL}_3(\mathbb{R})$. *Any finite subgroup of* $\mathrm{PGL}_3(\mathbb{R})$ *is either cyclic, or dihedral, or one of the symmetry groups of Platonic solids* $\mathfrak{A}_4$, $\mathfrak{S}_4$ *or* $\mathfrak{A}_5$.

Proposition 4.2. *Let X be a smooth real $\mathbb{R}$-rational surface with a conic bundle structure $\pi : X \to \mathbb{P}^1_{\mathbb{R}}$. Then*

$$\mathrm{J}(\mathrm{Aut}(X,\pi)) \leqslant 8.$$

Proof. Let $G \subset \mathrm{Aut}(X,\pi)$ be a finite group. Extending scalars to $\mathbb{C}$, we argue as in the proof of Proposition 3.3. The only difference is that G_F and G_B cannot be "exceptional" groups by Lemma 4.1 (1). So, one can find a normal abelian subgroup in G of index at most 8. □

We next consider real del Pezzo surfaces. For completeness sake, we also compute the weak Jordan constants for their automorphism groups. Note that for an algebraically closed field $\Bbbk$ of characteristic 0 one has $\bar{\mathrm{J}}(\mathrm{Aut}\, X) \leqslant 288$ for every smooth del Pezzo surface X over $\Bbbk$, see [PS16b, Corollary 3.2.5].

Proposition 4.3. *Let X be a smooth real $\mathbb{R}$-rational del Pezzo surface. Then one has*

$$\mathrm{J}(\mathrm{Aut}(X)) \leqslant 120, \quad \bar{\mathrm{J}}(\mathrm{Aut}(X)) \leqslant 20.$$

Proof. We again consider each $d = K_X^2$ separately. Since $\mathrm{J}(\mathrm{Aut}(X)) \leqslant \mathrm{J}(\mathrm{Aut}(X_{\mathbb{C}}))$, in most cases it will be enough to get a sharper bound for $\mathrm{J}(\mathrm{Aut}(X_{\mathbb{C}}))$, than in the proof of Proposition 3.1.

$d = 9$**:** Then $X \cong \mathbb{P}^2_{\mathbb{R}}$ and $\mathrm{J}(\mathrm{Aut}(X)) = |\mathfrak{A}_5| = 60$ by Lemma 4.1 (2). Clearly, $\bar{\mathrm{J}}(\mathrm{Aut}(X)) = 12$.

$d = 8$**:** If X is the blow up $\pi : X \to \mathbb{P}^2$ at one point, then every finite subgroup $G \subset \mathrm{Aut}(X)$ preserves the exceptional divisor of π isomorphic to $\mathbb{P}^1_{\mathbb{R}}$. So, we conclude by Lemma 4.1 (1). Now assume that $X_{\mathbb{C}} \cong \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$. Denote by $\mathrm{Q}_{r,s}$ the smooth quadric hypersurface

$$\{[x_1 : \ldots : x_{r+s}] : x_1^2 + \ldots + x_r^2 - x_{r+1}^2 - \ldots - x_{r+s}^2 = 0\} \subset \mathbb{P}^{r+s-1}_{\mathbb{R}}.$$

Then X is either $\mathrm{Q}_{3,1}$, or $\mathrm{Q}_{2,2}$. In the first case $\mathrm{Aut}(X) \cong \mathrm{PO}(3,1)$. Recall that

$$\mathrm{O}(3,1) = \mathrm{O}(3,1)^{\uparrow} \times \langle \pm I \rangle_2,$$

where I is the identity matrix and $\mathrm{O}(3,1)^{\uparrow}$ is the subgroup preserving the future light cone. The latter group is isomorphic to $\mathrm{PO}(3,1)$ and we may identify subgroups of $\mathrm{PO}(3,1)$ with subgroups of $\mathrm{O}(3,1)$. Using classification of finite subgroups of $\mathrm{O}(3,1)$ given in [PSA80], we see that every finite group $G \subset \mathrm{PO}(3,1)$ contains a normal abelian subgroup of index at most 60 and abelian (not necessarily normal) subgroup of index at most 12.

If $X \cong \mathrm{Q}_{2,2} \cong \mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$, then

$$\mathrm{Aut}(X) \cong \big(\mathrm{PGL}_2(\mathbb{R}) \times \mathrm{PGL}_2(\mathbb{R})\big) \rtimes \mathbb{Z}/2,$$

and the assertion easily follows from Lemma 4.1 (1).

$d = 7$: Then X is a blow up $\pi : X \to \mathbb{P}^2$ at two points. One of (-1)-curves on X is always defined over $\mathbb{R}$ and $\mathrm{Aut}(X)$-invariant, so we again conclude by Lemma 4.1 (1).

$d = 6$: One has $\bar{\mathrm{J}}(\mathrm{Aut}(X)) \leqslant \mathrm{J}(\mathrm{Aut}(X)) \leqslant \mathrm{J}(\mathrm{Aut}(X_{\mathbb{C}})) \leqslant |\mathrm{D}_6| = 12$.

$d = 5$: Then $\mathrm{Aut}(X_{\mathbb{C}}) \cong \mathfrak{S}_5$, so $\mathrm{J}(\mathrm{Aut}(X)) \leqslant 120$ and $\bar{\mathrm{J}}(\mathrm{Aut}(X)) \leqslant 20$. Note that there exists a real del Pezzo surface X of degree 5 with $\mathrm{Aut}(X) \cong \mathfrak{S}_5$ (it can be obtained by blowing up $\mathbb{P}^2_{\mathbb{R}}$ at 4 real points in general position). So, both bounds are sharp.

$d = 4$: As we already noticed, $\mathrm{Aut}(X_{\mathbb{C}}) \cong (\mathbb{Z}/2)^4 \rtimes \Gamma$, where $|\Gamma| \leqslant 10$. Thus,

$$\bar{\mathrm{J}}(\mathrm{Aut}(X)) \leqslant \mathrm{J}(\mathrm{Aut}(X)) \leqslant \mathrm{J}(\mathrm{Aut}(X_{\mathbb{C}})) \leqslant |\Gamma| \leqslant 10.$$

$d = 3$: From this moment we prefer to give more accurate bounds for $\mathrm{J}(\mathrm{Aut}(X))$. We will need these bounds in the proof of Theorem 1.11, although not all of them are needed in the present proof. One has the following possibilities for $\mathrm{Aut}(X_{\mathbb{C}})$ (see [Dol12, Theorem 9.5.8]):

- $|\mathrm{Aut}(X_{\mathbb{C}})| = 648$, $\mathrm{Aut}(X_{\mathbb{C}}) \cong (\mathbb{Z}/3)^3 \rtimes \mathfrak{S}_4$ and $X_{\mathbb{C}}$ is the Fermat cubic surface

$$\left\{[x_0 : x_1 : x_2 : x_3] : x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0\right\} \subset \mathbb{P}^3.$$

  Therefore, $\mathrm{J}(\mathrm{Aut}(X)) \leqslant |\mathfrak{S}_4| = 24$. Note that $\mathrm{Aut}(X) \cap (\mathbb{Z}/3)^3 \cong (\mathbb{Z}/3)^{\ell}$, where $\ell = 1, 2$, as $\mathrm{PGL}_4(\mathbb{R})$ does not contain $(\mathbb{Z}/3)^3$ (see e.g. [Yas16, Proposition 2.17]). Since every representation $\mathfrak{S}_4 \to \mathrm{GL}_{\ell}(\mathbb{F}_3)$ has non-trivial kernel, $\mathrm{Aut}(X)$ contains an abelian subgroup of index at most 12. We conclude that $\bar{\mathrm{J}}(\mathrm{Aut}(X)) \leqslant 12$.
- $|\mathrm{Aut}(X_{\mathbb{C}})| = 120$ and $\mathrm{Aut}(X_{\mathbb{C}}) \cong \mathfrak{S}_5$. Thus $\mathrm{J}(\mathrm{Aut}(X)) \leqslant \mathrm{J}(\mathfrak{S}_5) = 120$ and $\bar{\mathrm{J}}(\mathrm{Aut}(X)) \leqslant 20$.
- $|\mathrm{Aut}(X_{\mathbb{C}})| = 108$ and $\mathrm{Aut}(X_{\mathbb{C}}) \cong \mathscr{H}_3(3) \rtimes \mathbb{Z}/4$, where $\mathscr{H}_3(3)$ is the Heisenberg group of unipotent 3×3-matrices with entries in $\mathbb{F}_3$. Being a group of order 27, the Heisenberg group $\mathscr{H}_3(3)$ has non-trivial center, which must be a normal subgroup of $\mathrm{Aut}(X_{\mathbb{C}})$. Therefore, $\mathrm{J}(\mathrm{Aut}(X)) \leqslant 108/3 = 36$. On the other hand, $\mathscr{H}_3(3)$ contains an abelian subgroup of order 9, so $\bar{\mathrm{J}}(\mathrm{Aut}(X)) \leqslant 108/9 = 12$.
- $|\mathrm{Aut}(X_{\mathbb{C}})| = 54$ and $\mathrm{Aut}(X_{\mathbb{C}}) \cong \mathscr{H}_3(3) \rtimes \mathbb{Z}/2$. Similarly, one has $\bar{\mathrm{J}}(\mathrm{Aut}(X)) \leqslant 54/9 = 6$.
- $|\mathrm{Aut}(X_{\mathbb{C}})| \leqslant 24$. Then every non-trivial cyclic subgroup of $\mathrm{Aut}(X)$ has index at most 12.

$d = 2$: Recall that the anticanonical map $\psi_{|-K_X|} : X \to \mathbb{P}^2_{\mathbb{R}}$ is a double cover branched over a smooth quartic $B \subset \mathbb{P}^2_{\mathbb{R}}$. It is well known that $\mathrm{Aut}(X) \cong \mathrm{Aut}(B) \times \langle\gamma\rangle$, where γ is the Galois involution of the double cover (the *Geiser involution*). Since $\mathrm{Aut}(B) \subset \mathrm{PGL}_3(\mathbb{R})$ is a finite group, we can apply Lemma 4.1 (2). Namely, if $\mathrm{Aut}(B)$ is "exceptional", take $A = \langle\gamma\rangle$ as a desired normal abelian subgroup. If $\mathrm{Aut}(B) \cong \mathbb{Z}/n$, take $A = \mathrm{Aut}(X)$. If $\mathrm{Aut}(B) \cong \mathrm{D}_n$, take $A = C_n \times \langle\gamma\rangle$. In all the cases $\mathrm{J}(\mathrm{Aut}(X)) \leqslant 60$ and $\bar{\mathrm{J}}(\mathrm{Aut}(X)) \leqslant 12$.

$d = 1$: Let X be a real del Pezzo surface of degree 1. Recall that the linear system $|-K_X|$ is an elliptic pencil whose base locus consists of one point, which we denote by p. Clearly, $p \in X(\mathbb{R})$, so we have the natural faithful representation

$$\mathrm{Aut}(X) \to \mathrm{GL}(T_p X) \cong \mathrm{GL}_2(\mathbb{R}).$$

Thus $\mathrm{Aut}(X)$ is either cyclic, or dihedral, and $\mathrm{J}(\mathrm{Aut}(X)) \leqslant 2$. □

Corollary 4.4 (Theorem 1.9)**.** *One has*

$$\mathrm{J}(\mathrm{Cr}_2(\mathbb{R})) = 120, \quad \bar{\mathrm{J}}(\mathrm{Cr}_2(\mathbb{R})) = 20.$$

Proof. Let $G \subset \mathrm{Cr}_2(\mathbb{R})$ be a finite subgroup. By [DI09b, Theorem 5], we may assume that G acts biregularly on a smooth real $\mathbb{R}$-rational surface, which is either a del Pezzo surface, or a G-equivariant conic bundle. From Propositions 4.2 and 4.3, one gets $\mathrm{J}(\mathrm{Cr}_2(\mathbb{R})) \leqslant 120$ and $\bar{\mathrm{J}}(\mathrm{Cr}_2(\mathbb{R})) = 20$. The equalities are given by the group $\mathfrak{S}_5$, which occurs as the automorphism group of a real del Pezzo surface, obtained by blowing up $\mathbb{P}^2_{\mathbb{R}}$ at four real points in general position. □

Corollary 4.5 (Theorem 1.10)**.** *One has*

$$\mathrm{J}(\mathrm{Cr}_2(\mathbb{Q})) = 120, \quad \bar{\mathrm{J}}(\mathrm{Cr}_2(\mathbb{Q})) = 20.$$

Proof. Clearly, $\mathrm{J}(\mathrm{Cr}_2(\mathbb{Q})) \leqslant \mathrm{J}(\mathrm{Cr}_2(\mathbb{R}))$, $\bar{\mathrm{J}}(\mathrm{Cr}_2(\mathbb{Q})) \leqslant \bar{\mathrm{J}}(\mathrm{Cr}_2(\mathbb{R}))$. Since $\mathfrak{S}_5$ can be realized as the automorphism group of a degree 5 del Pezzo surface over $\mathbb{Q}$, we are done. □

Corollary 4.6 (Theorem 1.11)**.** *One has*

$$\mathrm{J}\big(\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))\big) = 60, \quad \bar{\mathrm{J}}\big(\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))\big) = 12,$$
$$\mathrm{J}(\mathrm{Q}_{3,1}(\mathbb{R})) = 60, \quad \bar{\mathrm{J}}(\mathrm{Q}_{3,1}(\mathbb{R})) = 12.$$

Proof. Take a finite subgroup $G \subset \mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))$ and regularize its action on some $\mathbb{R}$-rational surface X. Since we want $X(\mathbb{R})$ to be homeomorphic to $\mathbb{R}\mathbb{P}^2$, we may assume by [Kol97, Corollary 3.4.] that X is isomorphic to $\mathbb{P}^2_{\mathbb{R}}$ blown up at k pairs of complex conjugate points, where $k = 0, \ldots, 4$ and $d = K_X^2 = 9-2k$. From the proof of Proposition 4.3, one easily gets that $\mathrm{J}(\mathrm{Aut}(X)) \leqslant 60$ and $\bar{\mathrm{J}}(\mathrm{Aut}(X)) \leqslant 12$ in all the cases, except $d = k = 3$, $\mathrm{Aut}(X_{\mathbb{C}}) \cong \mathfrak{S}_5$, and $d = 5$, $k = 2$, $\mathrm{Aut}(X_{\mathbb{C}}) \cong \mathfrak{S}_5$. To conclude that $\mathrm{J}(\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))) = 60$ and $\bar{\mathrm{J}}(\mathrm{Aut}(\mathbb{P}^2(\mathbb{R}))) = 12$, it suffices to show that $\mathfrak{S}_5$ cannot occur as the automorphism group of such real surfaces.

If $d = 5$, then $\mathfrak{S}_5$ is the automorphism group of the Petersen graph of (-1)-curves on $X_{\mathbb{C}}$. In our case there are only 2 real lines on X, so $\mathrm{Aut}(X)$ cannot be isomorphic to $\mathfrak{S}_5$.

Assume that $d = k = 3$ and $\mathrm{Aut}(X) \cong \mathfrak{S}_5$. It is easy to see that there are exactly 3 real lines on X. Let $\tau \in \mathrm{Aut}(X)$ be of order 5. Then τ fixes each real line on X. Choose coordinates $[x_0 : \ldots : x_3]$ in $\mathbb{P}^3_{\mathbb{C}}$ such that one of these lines is given by $x_0 = x_1 = 0$. Then the equation of $X_{\mathbb{C}}$ has the form

$$x_0 q_1(x_2, x_3) + x_1 q_2(x_2, x_3) + \sum_{i+j=2} x_0^i x_1^j a_{ij}(x_2, x_3) + f(x_0, x_1) = 0,$$

where $\deg f = 3$. The quadratic forms q_1 and q_2 must be invariant with respect to an automorphism of order 5 of the line $x_2 = x_3 = 0$. Thus $q_1 = q_2 = 0$ and X is singular, so we get a contradiction with $\mathrm{Aut}(X) \cong \mathfrak{S}_5$.

Similarly, given a finite subgroup $G \subset \mathrm{Aut}(\mathrm{Q}_{3,1}(\mathbb{R}))$, we may assume that G acts biregularly either on a smooth $\mathbb{R}$-rational conic bundle X, or on an $\mathbb{R}$-rational del Pezzo surface X. In the former case we are done by Proposition 4.2. In the latter case, to preserve the real locus structure, the degree of X should be 8, 6, 4, or 2. From the proof of Proposition 4.3, we see that $\mathrm{J}(\mathrm{Aut}(X)) \leqslant 60$ and $\bar{\mathrm{J}}(\mathrm{Aut}(X)) \leqslant 12$ in these cases. As usual, the equality is given by the group $\mathfrak{A}_5$ acting on $\mathrm{Q}_{3,1}$. $\square$

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