Centralizers of elements of infinite order in plane Cremona groups

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Abstract

Let $\mathbf{K}$ be an algebraically closed field. The Cremona group $\mathrm{Cr}_2(\mathbf{K})$ is the group of birational transformations of the projective plane $\mathbb{P}^2_{\mathbf{K}}$. We carry out an overall study of centralizers of elements of infinite order in $\mathrm{Cr}_2(\mathbf{K})$ which leads to a classification of embeddings of $\mathbf{Z}^2$ into $\mathrm{Cr}_2(\mathbf{K})$, as well as a classification of maximal non-torsion abelian subgroups of $\mathrm{Cr}_2(\mathbf{K})$.

Contents

1 Introduction

Let $\mathbf{K}$ be an algebraically closed field. The *plane Cremona group* $\mathrm{Cr}_2(\mathbf{K})$ is the group of birational transformations of the projective plane $\mathbb{P}^2_{\mathbf{K}}$. It is isomorphic to the group of $\mathbf{K}$-algebra automorphisms of $\mathbf{K}(X_1,X_2)$, the function field of $\mathbb{P}^2_{\mathbf{K}}$. Using a system of homogeneous coordinates $[x_0;x_1;x_2]$, a birational transformation $f \in \mathrm{Cr}_2(\mathbf{K})$ can be written as

$$[x_0 : x_1 : x_2] \dashrightarrow [f_0(x_0,x_1,x_2) : f_1(x_0,x_1,x_2) : f_2(x_0,x_1,x_2)]$$

where f_0, f_1, f_2 are homogeneous polynomials of the same degree without common factor. This degree does not depend on the system of homogeneous coordinates. We call it the *degree* of f and denote it by $deg(f)$. Geometrically it is the degree of the pull-back by f of a general projective line. Birational transformations of degree 1 are homographies and form $\mathrm{Aut}(\mathbb{P}^2_{\mathbf{K}}) = \mathrm{PGL}_3(\mathbf{K})$, the group of automorphisms of the projective plane.

Four types of elements. Following the work of M.H. Gizatullin, S. Cantat, J. Diller and C. Favre, we can classify an element $f \in \mathrm{Cr}_2(\mathbf{K})$ into exactly one of the four following types according to the growth of the sequence $(deg(f^n))_{n\in\mathbf{N}}$ (The standard reference [DF01] is written for $\mathbf{K} = \mathbf{C}$ but it is known that the same proof works over an algebraically closed field $\mathbf{K}$ of characteristic different from 2 and 3. The only problem with characteristics 2 and 3 is that the important ingredient [Giz80] does not deal with quasi-elliptic fibrations. This minor issue has been clarified in [CD12a] and [CGL] so that the following classification holds for arbitrary characteristic.):

1. The sequence $(deg(f^n))_{n\in\mathbf{N}}$ is bounded, f is birationally conjugate to an automorphism of a rational surface X and a positive iterate of f lies in the connected component of the identity of the automorphism group $\mathrm{Aut}(X)$. We call f an *elliptic* element.

2. The sequence $(deg(f^n))_{n\in\mathbf{N}}$ grows linearly, f preserves a unique pencil of rational curves and f is not conjugate to an automorphism of any rational surface. We call f a *Jonquières twist*.

3. The sequence $(deg(f^n))_{n\in\mathbf{N}}$ grows quadratically, f is conjugate to an automorphism of a rational surface preserving a unique elliptic fibration. We call f a *Halphen twist*.

4. The sequence $(deg(f^n))_{n\in\mathbf{N}}$ grows exponentially and f is called *loxodromic*.

The Jonquières group Fix an affine chart of $\mathbb{P}^2$ with coordinates (x,y). *The Jonquières group* $\mathrm{Jonq}(\mathbf{K})$ is the subgroup of the Cremona group of all transformations of the form

$$(x,y) \dashrightarrow \left(\frac{ax+b}{cx+d}, \frac{A(x)y+B(x)}{C(x)y+D(x)}\right), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(\mathbf{K}), \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{PGL}_2(\mathbf{K}(x)).$$

In other words, $\mathrm{Jonq}(\mathbf{K})$ is the group of all birational transformations of $\mathbb{P}^1 \times \mathbb{P}^1$ permuting the fibres of the projection onto the first factor; it is isomorphic to the semidirect product $\mathrm{PGL}_2(\mathbf{K}) \ltimes \mathrm{PGL}_2(\mathbf{K}(x))$. A different choice of the affine chart yields a conjugation by an element of $\mathrm{PGL}_3(\mathbf{K})$. More generally a conjugation by an element of the Cremona group yields a group preserving a pencil of rational curves; conversely any two such groups are conjugate in $\mathrm{Cr}_2(\mathbf{K})$. Elements of $\mathrm{Jonq}(\mathbf{K})$ are either elliptic or Jonquières twists. We denote by $\mathrm{Jonq}_0(\mathbf{K})$ the normal subgroup of $\mathrm{Jonq}(\mathbf{K})$ that preserves fibrewise the rational fibration, i.e. the subgroup of those transformations of the form $(x,y) \dashrightarrow \left(x, \frac{A(x)y+B(x)}{C(x)y+D(x)}\right)$; it is isomorphic to $\mathrm{PGL}_2(\mathbf{K}(x))$. A Jonquières twist of the de Jonquières group will be called a *base-wandering Jonquières twist* if its action on the base of the rational fibration is of infinite order.

If $\mathbf{K} = \overline{\mathbf{F}_p}$ is the algebraic closure of a finite field, then $\mathbf{K}, \mathbf{K}^*$ and $\mathrm{PGL}_2(\mathbf{K})$ are all torsion groups. Thus, if $\mathbf{K} = \overline{\mathbf{F}_p}$ then base-wandering Jonquières twists do not exist. Whenever $\mathrm{char}(\mathbf{K}) = 0$, or $\mathrm{char}(\mathbf{K}) = p > 0$ and $\mathbf{K} \neq \overline{\mathbf{F}_p}$, there exist base-wandering Jonquières twists.

The group of automorphisms of a Hirzebruch surface will be systematically considered as a subgroup of the Jonquières group in the following way:

$$\mathrm{Aut}(\mathbb{F}_n) = \left\{ (x,y) \dashrightarrow \left(\frac{ax+b}{cx+d}, \frac{y+t_0+t_1x+\cdots+t_nx^n}{(cx+d)^n} \right) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{K}), t_0, \cdots, t_n \in \mathbf{K} \right\}.$$

Main results.

Theorem 1.1 *Let $f \in \mathrm{Cr}_2(\mathbf{K})$ be an element of infinite order. If the centralizer of f is not virtually abelian, then f is an elliptic element and a power of f is conjugate to an automorphism of $\mathbb{A}^2$ of the form $(x,y) \mapsto (x, y+1)$ or $(x,y) \mapsto (x, \beta y)$ with $\beta \in \mathbf{K}^*$.*

Theorem 1.2 *Let Γ be a subgroup of $\mathrm{Cr}_2(\mathbf{K})$ which is isomorphic to $\mathbf{Z}^2$. Then Γ has a pair of generators (f,g) such that one of the following (mutually exclusive) situations happens up to conjugation in $\mathrm{Cr}_2(\mathbf{K})$:*

1. *f,g are elliptic elements and $\Gamma \subset \mathrm{Aut}(X)$ where X is a rational surface;*
2. *f,g are Halphen twists which preserve the same elliptic fibration on a rational surface X, and $\Gamma \subset \mathrm{Aut}(X)$;*
3. *one or both of the f,g are Jonquières twists, and there exist $m,n \in \mathbf{N}^*$ such that the finite index subgroup of Γ generated by f^m and g^n is in an 1-dimensional torus over $\mathbf{K}(x)$ in $\mathrm{Jonq}_0(\mathbf{K}) = \mathrm{PGL}_2(\mathbf{K}(x))$;*
4. *f is a base-wandering Jonquières twist and g is elliptic. In some affine chart, we can write f,g in one of the following forms:*
   - *g is $(x,y) \mapsto (\alpha x, \beta y)$ and f is $(x,y) \dashrightarrow (\eta(x), yR(x^k))$ where $\alpha, \beta \in \mathbf{K}^*, \alpha^k = 1, R \in \mathbf{K}(x), \eta \in \mathrm{PGL}_2(\mathbf{K}), \eta(\alpha x) = \alpha\eta(x)$ and η is of infinite order;*
   - *(only when $\mathrm{char}(\mathbf{K}) = 0$) g is $(x,y) \mapsto (\alpha x, y+1)$ and f is $(x,y) \dashrightarrow (\eta(x), y + R(x))$ where $\alpha \in \mathbf{K}^*, R \in \mathbf{K}(x), R(\alpha x) = R(x), \eta \in \mathrm{PGL}_2(\mathbf{K}), \eta(\alpha x) = \alpha\eta(x)$ and η is of infinite order.*

Remark 1.3 When K is the algebraic closure of a finite field, the above list can be shortened since there is no elliptic elements of infinite order nor base-wandering Jonquières twists.

Remark 1.4 From Theorem 1.2 it is easy to see that (we will give a proof), when Γ is isomorphic to $\mathbf{Z}^2$, the degree function $deg : \Gamma \to \mathbf{N}$ is governed by the word length function with respect to some generators in the following sense. In the first case of the above theorem it is bounded. In the second case it is up to a bounded term a positive definite quadratic form over $\mathbf{Z}^2$. In the third case, if f is elliptic then deg is up to a bounded term $f^i \circ g^j \mapsto c|j|$ for some $c \in \mathbf{Q}_+$; otherwise we can choose two generators f_0, g_0 of $\Gamma \cap \mathrm{Jonq}_0(\mathbf{K})$ such that deg restricted to $\Gamma \cap \mathrm{Jonq}_0(\mathbf{K})$ is up to a bounded term $f_0^i \circ g_0^j \mapsto c_1|i| + c_2|j|$ for some $c_1, c_2 \in \mathbf{Q}_+$. In the fourth case the degree function is up to a bounded term $f^i \circ g^j \mapsto c|i|$ for some $c \in \mathbf{Q}_+$. Note that if f and g are two Jonquières twists of $\mathrm{Jonq}(\mathbf{K})$ that do not necessarily commute, then the degree of $f^i \circ g^j$ is always dominated by $deg(f)|i| + deg(g)|j|$ (see Lemma 5.7 [BC16]).

A direct corollary of Theorem 1.2 is:

Corollary 1.5 *Let $G \subset \mathrm{Cr}_2(\mathbf{K})$ be a subgroup isomorphic to $\mathbf{Z}^2$. If G is not an elliptic subgroup then there exists a non-trivial element of G which preserves each member of a pencil of rational or elliptic curves.*

Theorem 1.2 is based on several known results. The main new feature is the fourth case. We reformulate this special case as a corollary (see Theorem 3.2 for a more precise reformulation):

Corollary 1.6 *Let $G \subset \mathrm{Jonq}(\mathbf{K})$ be a subgroup isomorphic to $\mathbf{Z}^2$. Suppose that the action of G on the base of the rational fibration is faithful. Then G is an elliptic subgroup.*

A *maximal abelian subgroup* is an abelian subgroup which is not strictly contained in any other abelian subgroup. Over the field of complex numbers, finite abelian subgroups of $\mathrm{Cr}_2(\mathbf{C})$ have been classified in [Bla07]. We will use Theorem 1.2 to classify maximal abelian subgroups of $\mathrm{Cr}_2(\mathbf{K})$ which contain at least one element of infinite order, see Theorem 4.1.

Previously known results. Let us begin with the group of polynomial automorphism of the affine plane $\mathrm{Aut}(\mathbb{A}^2)$. It can be seen as a subgroup of $\mathrm{Cr}_2(\mathbf{K})$. It is the amalgamated product of the group of affine automorphisms with the so called *elementary group*

$$\mathrm{El}(\mathbf{K}) = \{(x,y) \mapsto (\alpha x + \beta, \gamma y + P(x)) | \alpha, \beta, \gamma \in \mathbf{K}, \alpha\beta \neq 0, P \in \mathbf{K}[x]\}.$$

Let $\mathbf{K}$ be the field of complex numbers. S. Friedland and J. Milnor showed in [FM89] that an element of $\mathrm{Aut}(\mathbf{C}^2)$ is either conjugate to an element of $\mathrm{El}(\mathbf{K})$ or to a gengeralized Hénon map, i.e. a composition $f_1 \circ \cdots \circ f_n$ where the f_i are Hénon maps of the form $(x,y) \mapsto (y, P_i(y) - \delta_i x)$ with $\delta_i \in \mathbf{C}^*$, $P_i \in \mathbf{C}[y]$, $deg(P_i) \geq 2$. S. Lamy showed in [Lam01] that the centralizer in $\mathrm{Aut}(\mathbf{C}^2)$ of a generalized Hénon map is finite by cyclic, and that of an element of $\mathrm{El}(\mathbf{C})$ is uncountable (see also [Bis04]). Note that, when viewed as elements of $\mathrm{Cr}_2(\mathbf{C})$, a generalized Hénon map is loxodromic and an element of $\mathrm{El}(\mathbf{C})$ is elliptic.

As regards the Cremona group, centralizers of loxodromic elements are known to be finite by cyclic (S. Cantat [Can11], J. Blanc-S. Cantat [BC16]). Centralizers of

Halphen twists are virtually abelian of rank at most 8 (M.K. Gizatullin [Giz80], S. Cantat [Can11]). When $\mathbf{K}$ is the field of complex numbers, centralizers of elliptic elements of infinite order are completely described by J. Blanc-J. Déserti in [BD15] and centralizers of Jonquières twists in $\mathrm{Jonq}_0(\mathbf{K})$ are completely described by D. Cerveau-J. Déserti in [CD12b]. Centralizers of base-wandering Jonquières twists are also studied in [CD12b] but they were not fully understood, for example the results in loc. cit. are not sufficient for classifying pairs of Jonquières twists generating a copy of $\mathbf{Z}^2$. Thus, in order to obtain a classification of embeddings of $\mathbf{Z}^2$ in $\mathrm{Cr}_2(\mathbf{K})$, we need a detailed study of centralizers of base-wandering Jonquières twists, which is the main task of this article. Regarding the elements of finite order and their centralizers in $\mathrm{Cr}_2(\mathbf{K})$, the problem is of a rather different flavour and we refer the readers to [Bla07], [DI09], [Ser10], [Ure18] and the references therein.

Remark 1.7 There is a topology on $\mathrm{Cr}_2(\mathbf{K})$, called Zariski toplogy, which is introduced by M. Demazure and J-P. Serre in [Dem70] and [Ser10]. Note that the Zariski topology does not make $\mathrm{Cr}_2(\mathbf{K})$ an infinite dimensional algebraic group (cf. [BF13]). With respect to the Zariski topology, the centralizer of any element of $\mathrm{Cr}_2(\mathbf{K})$ is closed (J-P. Serre [Ser10]). When K is a local field, J. Blanc and J-P. Furter construct in [BF13] an Euclidean topology on $\mathrm{Cr}_2(\mathbf{K})$ which when restricted to $\mathrm{PGL}_3(\mathbf{K})$ coincides with the Euclidean topology of $\mathrm{PGL}_3(\mathbf{K})$; centralizers are also closed with respect to the Euclidean toplogy. In particular the intersection of the centralizer of an element in $\mathrm{Cr}_2(\mathbf{K})$ with an algebraic subgroup G of $\mathrm{Cr}_2(\mathbf{K})$ is a closed subgroup of G, with respect to the Zariski topology of G (and with respect to the Euclidean topology when the later is present).

Comparison with other results. S.Smale asked in the '60s if, in the group of diffeomorphisms of a compact manifold, the centralizer of a generic diffeomorphism consists only of its iterates. There has been a lot of work on this question, see for example [BCW09] for an affirmative answer in the C^1 case. Similar phenomenons also appear in the group of germs of 1-dimensional holomorphic diffeomorphisms at $0 \in \mathbf{C}$ ([É81]). See the introduction of [CD12b] for more references in this direction. With regard to $\mathrm{Cr}_2(\mathbf{K})$, it is known that loxodromic elements form a Zariski dense subset of $\mathrm{Cr}_2(\mathbf{K})$ (cf. [Xie15], [BD05]) and that their centralizers coincide with the cyclic group formed by their iterates up to finite index (cf. [BC16]). Centralizers of general Jonquières twists are also finite by cyclic (Remark 3.5).

One may compare our classification of $\mathbf{Z}^2$ in $\mathrm{Cr}_2(\mathbf{K})$ to the following two theorems where the situations are more rigid. The first can be seen as a continuous counterpart and is proved by F. Enriques [Enr93] and M. Demazure [Dem70], the second can be seen as a torsion counterpart and is proved by A. Beauville [Bea07]:

1. *If* $\mathbf{K}^{*r}$ *embeds as an algebraic subgroup into* $\mathrm{Cr}_2(\mathbf{K})$*, then* $r \leq 2$*; if* $r = 2$ *then the embedding is conjugate to an embedding into the group of diagonal matrices* Δ *in* $\mathrm{PGL}_3(\mathbf{K})$.

2. *If* $p \geq 5$ *is a prime number different from the characteristic of* $\mathbf{K}$ *and if* $(\mathbf{Z}/p\mathbf{Z})^r$ *embeds into* $\mathrm{Cr}_2(\mathbf{K})$*, then* $r \leq 2$*; if* $r = 2$ *then the embedding is conjugate to an embedding into the group of diagonal matrices* Δ *in* $\mathrm{PGL}_3(\mathbf{K})$.

The classification of $\mathbf{Z}^2$ in $\mathrm{Cr}_2(\mathbf{K})$ is a very natural special case of the study of finitely generated subgroups of $\mathrm{Cr}_2(\mathbf{K})$; and information on centralizers can be useful for studying homomorphisms from other groups into $\mathrm{Cr}_2(\mathbf{K})$, see for example [D0́6]. We refer the reader to the surveys [Fav10],[Can18] for representations of finitely generated groups into $\mathrm{Cr}_2(\mathbf{K})$ and [CX18] for general results in higher dimension.

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2 Elements which are not base-wandering Jonquières twists

This section contains a quick review of some scattered results about centralizers from [Can11],[BD15],[CD12b],[BC16]. Some of the proofs are reproduced, because the original proofs were written over $\mathbf{C}$ on the one hand, and because we will need some by-products of the proofs on the other hand.

2.1 Loxodromic elements

Theorem 2.1 ([BC16] Corollary 4.7) *Let $f \in \mathrm{Cr}_2(\mathbf{K})$ be a loxodromic element. The infinite cyclic group generated by f is a finite index subgroup of the centralizer of f in* $\mathrm{Cr}_2(\mathbf{K})$.

Proof We provide a proof which is simpler than [BC16]. The Cremona group $\mathrm{Cr}_2(\mathbf{K})$ acts faithfully by isometries on an infinite dimensional hyperbolic space $\mathbb{H}$ and the action of a loxodromic element is loxodromic in the sense of hyperbolic geometry (see [Can11], [Can18]). In particular there is an f-invariant geodesic $Ax(f)$ on which f acts by translation and the translation length is $\log(\lim_{n\to\infty} deg(f^n)^{1/n})$. The centralizer $\mathrm{Cent}(f)$ preserves $Ax(f)$ and by considering translation lengths we get a morphism $\phi : \mathrm{Cent}(f) \to \mathbf{R}$. We claim that the image of ϕ is discrete thus cyclic. Let us see first how the conclusion follows from the claim. Let $x \in \mathbb{H}$ be a point which corresponds to an ample class and let y be an arbitrary point on $Ax(f)$. Since the kernel $\mathrm{Ker}(\phi)$ fixes $Ax(f)$ pointwise, for any element g of $\mathrm{Ker}(\phi)$ the distance $d(x, g(x))$ is bounded by $2d(x,y)$. This implies that $\mathrm{Ker}(\phi)$ is a subgroup of $\mathrm{Cr}_2(\mathbf{K})$ of bounded degree. If $\mathrm{Ker}(\phi)$ were infinite then its Zariski closure G in $\mathrm{Cr}_2(\mathbf{K})$ would be an algebraic subgroup of strictly positive dimension contained, after conjugation, in the automorphism group of a rational surface. As $\mathrm{Cent}(f)$ is Zariski closed, the elements of G commute with f. The orbits of a one-parameter subgroup of G would form an f-invariant pencil of curves. This contradicts the fact that f is loxodromic. Consequently $\mathrm{Ker}(\phi)$ is finite and hence $\mathrm{Cent}(f)$ is finite by cyclic.

Now let us prove the claim that the image of ϕ is discrete. This follows directly from a spectral gap property for translation lengths of loxodromic elements proved in

[BC16]. We give here an easier direct proof found with S. Cantat. Suppose by contradiction that there is a sequence $(g_n)_n$ of distinct elements of $\mathrm{Cent}(f)$ whose translation lengths on $Ax(f)$ tend to 0 when n goes to infinity. Without loss of generality, we can suppose the existence of a point y on $Ax(f)$ and a real number $\varepsilon > 0$ such that $\forall n, d(y, g_n(y)) < \varepsilon$. Let $x \in \mathbb{H}$ be an element which corresponds to an ample class. Then it follows that

$$\forall n, d(x, g_n(x)) \leq d(x,y) + d(y, g_n(y)) + d(g_n(y), g_n(x)) < 2d(x,y) + \varepsilon =: d,$$

i.e. the sequence $(g_n)_n$ is of bounded degree d. Elements of degree less than d of the Cremona group form a quasi-projective variety $\mathrm{Cr}_2^d(\mathbf{K})$. JunYi Xie proved in [Xie15] that for any $0 < \lambda < \log(d)$, the loxodromic elements of $\mathrm{Cr}_2^d(\mathbf{K})$ whose translation lengths are greater than λ form a Zariski open dense subset of $\mathrm{Cr}_2^d(\mathbf{K})$. Thus the g_n give rise to a strictly ascending chain of Zariski open subsets of $\mathrm{Cr}_2^d(\mathbf{K})$, contradicting the noetherian property of Zariski topology. This finishes the proof. Note that [Xie15] is also used to prove the spectral gap property in [BC16]. □

2.2 Halphen twists

We only recall here the final arguments of the proofs.

Theorem 2.2 ([Giz80] and [Can11] Proposition 4.7) *Let $f \in \mathrm{Cr}_2(\mathbf{K})$ be a Halphen twist. The centralizer $\mathrm{Cent}(f)$ of f in $\mathrm{Cr}_2(\mathbf{K})$ contains a finite index abelian subgroup of rank less than or equal to 8.*

Proof Being a Halphen twist, the birational transformation f is up to conjugation an automorphism of a rational surface and preserves a relatively minimal elliptic fibration. This f-invariant fibration is unique. As a consequence $\mathrm{Cent}(f)$ acts by automorphisms preserving this fibration. It is proved in [Giz80] (see [CGL] for a clarification in characteristics 2 and 3) that the automorphism group of a rational minimal elliptic surface has a finite index abelian subgroup of rank less than 8. □

2.3 Elliptic elements of infinite order

In this section we reproduce a part of [BD15]; we follow the original proofs (for $\mathrm{char}(\mathbf{K}) = 0$) in loc. cit. and some extra details are added in case $\mathrm{char}(\mathbf{K}) > 0$.

We omit the proof of the following key proposition which is based on a G-Mori-program for rational surfaces due to J. Manin [Man67] and V. Iskovskih [Isk79].

Proposition 2.3 ([BD15] Proposition 2.1) *Let S be a smooth rational surface over $\mathbf{K}$. Let $f \in \mathrm{Aut}(S)$ be an automorphism of infinite order whose action on $\mathrm{Pic}(S)$ is of finite order. Then there exists a birational morphism $S \to X$ where X is a Hirzebruch surface $\mathbb{F}_n$ ($n \neq 1$) or the projective plane $\mathbb{P}^2$, which conjugates f to an automorphism of X.*

Proposition 2.4 ([BD15] Proposition 2.3) *Let $f \in \mathrm{Cr}_2(\mathbf{K})$ be an elliptic element of infinite order. Then f is conjugate to an automorphism of $\mathbb{P}^2$. Furthermore there exists an affine chart with affine coordinates (x,y) on which f acts by automorphism of the following form:*

1. $(x,y) \mapsto (\alpha x, \beta y)$ *where* $\alpha, \beta \in \mathbf{K}^*$ *are such that the kernel of the group homomorphism* $\mathbf{Z}^2 \to \mathbf{K}^*, (i,j) \mapsto \alpha^i \beta^j$ *is generated by* $(k,0)$ *for some* $k \in \mathbf{Z}$*;*

2. $(x,y) \mapsto (\alpha x, y+1)$ *where* $\alpha \in \mathbf{K}^*$ *and* α *is of infinite order if* $\mathrm{char}(\mathbf{K}) > 0$.

Remark 2.5 If $\mathbf{K} = \overline{\mathbf{F}_p}$ then every elliptic element is of finite order.

As a byproduct of the proof of Proposition 2.4, we will get the following:

Proposition 2.6 *Let* f *be an automorphism of a Hirzebruch surface which preserves the rational fibration fibre by fibre (we do not assume that* f *is of infinite order). Then there exists an affine chart on which* f *acts as an automorphism of the following form:*

1. $(x,y) \mapsto (x, \beta y)$ *where* $\beta \in \mathbf{K}^*$*;*

2. $(x,y) \mapsto (x, y+1)$.

Here x *is the coordinate on the base of the rational fibration.*

Proof (of Proposition 2.4) Proposition 2.3 says that f is conjugate to an automorphism of $\mathbb{P}^2$ or of a Hirzebruch surface.

Let's consider first the case when $f \in \mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}_3(\mathbf{K})$. By putting the corresponding matrix in Jordan normal form, we can find an affine chart on which f is, up to conjugation, of one of the following form: 1) $(x,y) \mapsto (\alpha x, \beta y)$; 2) $(x,y) \mapsto (\alpha x, y+1)$; 3) $(x,y) \mapsto (x+y, y+1)$. If $\mathrm{char}(\mathbf{K}) > 0$ then f can not be of the third form since it would be of finite order; if $\mathrm{char}(\mathbf{K}) = 0$ then in the third case f is conjugate by $[x:y:z] \dashrightarrow [xz - \frac{1}{2}y(y-z) : yz : z^2]$ to $(x,y) \mapsto (x, y+1)$. We now show that in the first case α, β can be chosen to verify the conditon in the proposition. Let $\phi : (x,y) \mapsto (\alpha x, \beta y)$ be a diagonal automorphism, we denote by $\Delta(\phi)$ the kernel of the group morphism $\mathbf{Z}^2 \to \mathbf{K}^*, (i,j) \mapsto \alpha^i \beta^j$. For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z})$, we denote by $M(\phi)$ the diagonal automorphism $(x,y) \mapsto (\alpha^a \beta^b x, \alpha^c \beta^d y)$, i.e. the conjugate of ϕ by the monomial map $(x,y) \dashrightarrow (x^a y^b, x^c y^d)$. We have the relation $\Delta(M(\phi)) = (M^\intercal)^{-1}(\Delta(\phi))$. This implies that up to conjugation by a monomial map we can suppose that our elliptic element f satisfies $\Delta(f) = <(k_1, 0), (0, k_1 k_2)>$ where $k_1, k_2 \in \mathbf{Z}$. Since f is of infinite order, $k_1 k_2$ must be 0.

If $f \in \mathrm{Aut}(\mathbb{F}_0) = \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$, then we reduce to the case of $\mathbb{P}^2$ by blowing up a fixed point and contracting the strict transforms of the two rulings passing through the point. If $f \in \mathrm{Aut}(\mathbb{F}_n)$ for $n \geq 2$ and if f has a fixed point which is not on the exceptional section, then we can reduce to $\mathbb{F}_{n-1}$ by making an elementary transformation at the fixed point.

Suppose now that $f \in \mathrm{Aut}(\mathbb{F}_n), n \geq 2$ and its fixed points are all on the exceptional section. By removing the exceptional section and an invariant fibre of the rational fibration, we get an open subset isomorphic to $\mathbb{A}^2$ on which f can be written as: $(x,y) \mapsto (\alpha x, \beta y + Q(x))$ or $(x,y) \mapsto (x+1, \beta y + Q(x))$ where $\alpha, \beta \in \mathbf{K}^*$ and Q is a polynomial of degree $\leq n$.

In the first case, the fact that there is no extra fixed point on the fibre $x = 0$ implies $\beta = 1$ and $Q(0) \neq 0$. The action on the fibre at infinity can be obtained by a change of variables $(x',y')=(1/x,y/x^n)$, so the fact that there is no extra fixed point on it implies $\beta=\alpha^n$ and $\deg(Q)=n$. This forces α to be a primitive r-th root of unity for some $r\in\mathbf{N}$. Conjugating f by $(x,y)\mapsto(x,y+\gamma x^d)$, we replace $Q(x)$ with $Q(x)+\gamma(\alpha^d-1)x^d$. This allows us to eliminate the term x^d of Q unless $\alpha^d=1$. So we can assume that f is of the form $(x,y)\mapsto(\alpha x,y+\tilde{Q}(x^r))$ where $\alpha^r=1$ and $\tilde{Q}\in\mathbf{K}[x]$. Then f is conjugate to $(x,y)\mapsto(\alpha x,y+1)$ by $(x,y)\dashrightarrow(x,y/\tilde{Q}(x^r))$. Remark that this case does not happen in positive characteristic because an automorphism of this form would be of finite order. Note that in this paragraph we did not use the fact that f is of infinite order, so that Proposition 2.6 is proved.

Suppose now we are in the second case. There is no extra fixed point if and only if $\beta=1$ and $\deg(Q)=n$. If $\mathrm{char}(\mathbf{K})>0$ and if $\beta=1$, then f would be of finite order. Therefore we can assume $\mathrm{char}(\mathbf{K})=0$. In that case, we can decrease the degree of Q by conjugating f by a well chosen birational transformation of the form $(x,y)\dashrightarrow(x,y+\gamma x^{n+1})$ with $\gamma\in\mathbf{K}^*$. By induction we get $(x,y)\mapsto(x+1,y)$ at last. $\square$

Once we have the above normal forms, explicit calculations can be done:

Theorem 2.7 ([BD15] Lemmas 2.7 and 2.8) *Let $f\in\mathrm{Cr}_2(\mathbf{K})$ be an elliptic element of infinite order.*

1. *If f is of the form $(x,y)\mapsto(\alpha x,\beta y)$ where $\alpha,\beta\in\mathbf{K}^*$ are such that the kernel of the group homomorphism $\mathbf{Z}^2\to\mathbf{K}^*,(i,j)\mapsto\alpha^i\beta^j$ is generated by $(k,0)$ for some $k\in\mathbf{Z}$, then the centralizer of f in $\mathrm{Cr}_2(\mathbf{K})$ is*

   $$\mathrm{Cent}(f)=\{(x,y)\dashrightarrow(\eta(x),yR(x^k))|R\in\mathbf{K}(x),\eta\in\mathrm{PGL}_2(\mathbf{K}),\eta(\alpha x)=\alpha\eta(x)\}.$$

2. *If $\mathrm{char}(\mathbf{K})=0$ and if f is of the form $(x,y)\mapsto(\alpha x,y+1)$, then the centralizer of f in $\mathrm{Cr}_2(\mathbf{K})$ is*

   $$\mathrm{Cent}(f)=\{(x,y)\dashrightarrow(\eta(x),y+R(x))|\eta\in\mathrm{PGL}_2(\mathbf{K}),\eta(\alpha x)=\alpha\eta(x),R\in\mathbf{K}(x),R(\alpha x)=R(x)\}.$$

   If α is not a root of unity then R must be constant and $\eta(x)=\beta x$ for some $\beta\in\mathbf{K}^*$.

3. *If $\mathrm{char}(\mathbf{K})=p>0$ and if f is of the form $(x,y)\mapsto(\alpha x,y+1)$ (where α must be of infinite order), then the centralizer of f in $\mathrm{Cr}_2(\mathbf{K})$ is*

   $$\mathrm{Cent}(f)=\{(x,y)\dashrightarrow(R(y)x,y+t)|t\in\mathbf{K},R(y)=S(y)S(y-1)\cdots S(y-p+1),S\in\mathbf{K}(y)\}.$$

Remark 2.8

$$\{\eta\in\mathrm{PGL}_2(\mathbf{K})|\eta(\alpha x)=\alpha\eta(x)\}=\begin{cases}\mathrm{PGL}_2(\mathbf{K}) & \text{if}\quad \alpha=1\\ \{x\mapsto\gamma x^{\pm1}|\gamma\in\mathbf{K}^*\} & \text{if}\quad \alpha=-1\\ \{x\mapsto\gamma x|\gamma\in\mathbf{K}^*\} & \text{if}\quad \alpha\neq\pm1\end{cases}$$

Proof *First case.* We treat first the case where f is of the form $(x,y)\mapsto(\alpha x,\beta y)$. Let $(x,y)\dashrightarrow(\frac{P_1(x,y)}{Q_1(x,y)},\frac{P_2(x,y)}{Q_2(x,y)})$ be an element of $\mathrm{Cent}(f)$; here $P_1,P_2,Q_1,Q_2\in\mathbf{K}[x,y]$. The commutation relation gives us

$$\frac{P_1(\alpha x,\beta y)}{Q_1(\alpha x,\beta y)}=\frac{\alpha P_1(x,y)}{Q_1(x,y)},\quad \frac{P_2(\alpha x,\beta y)}{Q_2(\alpha x,\beta y)}=\frac{\beta P_2(x,y)}{Q_2(x,y)}$$

which imply that P_1,P_2,Q_1,Q_2 are eigenvectors of the $\mathbf{K}$-linear automorphism $\mathbf{K}[x,y]\to\mathbf{K}[x,y], g(x,y)\mapsto g(\alpha x,\beta y)$. Therefore each one of the P_1,P_2,Q_1,Q_2 is a product of a monomial in x,y with a polynomial in $\mathbf{K}[x^k]$. Then we must have $\frac{P_1(x,y)}{Q_1(x,y)}=xR_1(x^k)$ and $\frac{P_2(x,y)}{Q_2(x,y)}=yR_2(x^k)$ for some $R_1,R_2\in\mathbf{K}(x)$. The first factor $\frac{P_1(x,y)}{Q_1(x,y)}$ only depends on x, so for f to be birational it must be an element of $\mathrm{PGL}_2(\mathbf{K})$. The conclusion in this case follows.

Second case. We now treat the case where $\mathrm{char}(\mathbf{K})=0$ and where f is of the form $(x,y)\mapsto(\alpha x,y+1)$. Let $(x,y)\dashrightarrow(\frac{P_1(x,y)}{Q_1(x,y)},\frac{P_2(x,y)}{Q_2(x,y)})$ be an element of $\mathrm{Cent}(f)$. We have

$$\frac{P_1(\alpha x,y+1)}{Q_1(\alpha x,y+1)}=\frac{\alpha P_1(x,y)}{Q_1(x,y)}\quad\frac{P_2(\alpha x,y+1)}{Q_2(\alpha x,y+1)}=\frac{P_2(x,y)}{Q_2(x,y)}+1. \tag{1}$$

The first equation implies that P_1,Q_1 are eigenvectors of the $\mathbf{K}$-linear automorphism $\mathbf{K}[x,y]\to\mathbf{K}[x,y], g(x,y)\mapsto g(\alpha x,y+1)$. We view an element of $\mathbf{K}[x,y]$ as a polynomial in x with coefficients in $\mathbf{K}[y]$. Since the only eigenvector of the $\mathbf{K}$-linear automorphism $\mathbf{K}[y]\to\mathbf{K}[y], g(y)\mapsto g(y+1)$ is 1 (this is not true if $\mathrm{char}(\mathbf{K})>0$), we deduce that P_1,Q_1 depend only on x. Thus, $\frac{P_1(x,y)}{Q_1(x,y)}$ is an element η of $\mathrm{PGL}_2(\mathbf{K})$.

We derive $\psi=\frac{P_2}{Q_2}$ and get

$$\frac{\partial\psi}{\partial y}(\alpha x,y+1)=\frac{\partial\psi}{\partial y}(x,y),\quad \frac{\partial\psi}{\partial x}(\alpha x,y+1)=\alpha^{-1}\frac{\partial\psi}{\partial x}(x,y).$$

As before, this means that $\frac{\partial\psi}{\partial y},\frac{\partial\psi}{\partial x}$ only depend on x (not true if $\mathrm{char}(\mathbf{K})>0$). Hence, we can write ψ as $ay+B(x)$ with $a\in\mathbf{K}^*$ and $B\in\mathbf{K}(x)$. Then equation (1) implies $B(\alpha x)=B(x)+1-a$, which implies further $x\frac{\partial B}{\partial x}(x)$ is invariant under $x\mapsto\alpha x$. If α is of infinite order, then $\frac{\partial B}{\partial x}(x)=\frac{c}{x}$ for some constant $c\in\mathbf{K}$. This is only possible if $c=0$. So B is constant and $a=1$ in this case. If α is a primitive k-th root of unity, then $(\eta(x),ay+B(x))$ commutes with $f^k:(x,y)\mapsto(x,y+k)$. This yields $a=1$ and $B(\alpha x)=B(x)$.

Third case. We finally treat the case where $\mathrm{char}(\mathbf{K})=p>0$ and where f is of the form $(x,y)\mapsto(\alpha x,y+1)$ with α of infinite order. Let $g\in\mathrm{Cent}(f)$. Then g commutes with $f^p:(x,y)\mapsto(\alpha^p x,y)$ which is in the form of case 1 (the roles of x,y are exchanged). Thus, we know that g writes as $(A(y)x,\eta(y))$ where $\eta\in\mathrm{PGL}_2(\mathbf{K})$ and $A\in\mathbf{K}(x)$. Then $f\circ g=g\circ f$ implies that η is $y\mapsto y+R$ for some $R\in\mathbf{K}$ and that $A(y+1)=A(y)$. The last equation implies $A(y)=S(y)S(y-1)\cdots S(y-p+1)$ for some $S\in\mathbf{K}(x)$. $\square$

For later use, we determine when an element of the centralizers appeared in Theorem 2.7 is elliptic. Though we will use some of the materials of Section 3.1 in the proofs, we find it more natural to state these facts here.

Lemma 2.9 *Let $f:(x,y)\dashrightarrow(\eta(x),yR(x)),\eta\in\mathrm{PGL}_2(\mathbf{K}),R\in\mathbf{K}(x)$ be an elliptic element. Then*

1. *either $R\in\mathbf{K}$,*

2. *or* $R(x)=\frac{rS(x)}{S(\eta(x))}$ *with* $r\in\mathbf{K}^*$ *and* $S\in\mathbf{K}(x)\backslash\mathbf{K}$.

Proof If η is the identity, then we see easily, by looking at the degree growth, that f is elliptic if and only if R is constant.

From now on assume that η is not the identity. We claim that f is conjugate by an element of $\mathrm{Jonq}_0(\mathbf{K})$ to an automorphism of a Hirzebruch surface. By Corollary 3.8, this does not hold if and only if η is of finite order d and f^d is a Jonquières involution (see Corollary 3.9 for the terminology). However if η is of finite order d then f^d is of the form $(x,y)\dashrightarrow(\eta(x),y\tilde{R}(x))$ with $\tilde{R}(x)=R(x)\cdots R(\eta^{d-1}(x))$, which is never a Jonquières involution. This proves the claim.

By Theorem 3.6, the conjugation which turns f into an automorphism of a Hirzebruch surface is a sequence of elementary transformations. After conjugation it preserves the two strict transforms of the two sections $\{y=0\}$ and $\{y=\infty\}$. Therefore there exists $g\in\mathrm{Jonq}_0(\mathbf{K})$ of the form $(x,y)\dashrightarrow(x,yS(x)), S\in\mathbf{K}(x)$ such that $g\circ f\circ g^{-1}$ is $(x,y)\dashrightarrow(\eta(x),ry)$ with $r\in\mathbf{K}^*$. Hence f is $(x,y)\dashrightarrow(\eta(x),y\frac{rS(x)}{S(\eta(x))}$. □

Remark 2.10 In the above lemma S may not be unique. If η has finite order and $T\in\mathbf{K}(x)$ is such that $T(x)=T(\eta(x))$, then $\frac{S(x)}{S(\eta(x))}=\frac{T(x)S(x)}{T(\eta(x))S(\eta(x))}$.

Lemma 2.11 *Let* $f:(x,y)\dashrightarrow(\eta(x),y+R(x)),\eta\in\mathrm{PGL}_2(\mathbf{K}),R\in\mathbf{K}(x)$ *be an elliptic element. Then*

1. *either* η *has finite order,*
2. *or for a coordinate* x' *such that* η *is* $x'\mapsto x'+1$ *or* $x'\mapsto\nu x'$ *with* $\nu\in\mathbf{K}^*$, *R is a polynomial in* x'.

Proof It is clear that, if η has finite order then the degree of f^n is bounded for all $n\in\mathbf{Z}$. Assume that η has infinite order, then for some coordinate x', η writes as $\eta'(x')=x'\mapsto\nu x'+u$ with $\nu,u\in\mathbf{K}$. In coordinates (x',y), write the transformation f as $(x,y)\dashrightarrow(\eta'(x'),y+R'(x'))$ where $R'(x')=\frac{P(x')}{Q(x')}$ with $P,Q\in\mathbf{K}[x']$. For $n\in\mathbf{N}^*$, the iterate f^n is

$$(x,y)\dashrightarrow\left(\eta'(x'),y+\frac{P(x')}{Q(x')}+\cdots+\frac{P(\eta'^{n-1}(x'))}{Q(\eta'^{n-1}(x'))}\right).$$

If $Q\notin\mathbf{K}$, then the number of different factors of the polynomials $Q(x'),\cdots,Q(\eta'^{n-1}(x'))$ would go to infinity when n tends to infinity, which would imply that the degrees of the f^n are not bounded. Therefore for f to be elliptic, R' must be a polynomial. □

2.4 Jonquières twists with trivial action on the base

We follow [CD12b] in this section.

Lemma 2.12 *Let* $f\in\mathrm{Jonq}(\mathbf{K})$ *be a Jonquières twist. Let* $\mathrm{Cent}(f)$ *be the centralizer of* f *in* $\mathrm{Cr}_2(\mathbf{K})$. *Then* $\mathrm{Cent}(f)\subset\mathrm{Jonq}(\mathbf{K})$.

Proof The rational fibration preserved by a Jonquières twist f is unique, thus is also preserved by $\mathrm{Cent}(f)$. $\square$

Let us consider centralizers of Jonquières twists in $\mathrm{Jonq}_0(\mathbf{K}) = \mathrm{PGL}_2(\mathbf{K}(x))$ which is a linear algebraic group over the function field $\mathbf{K}(x)$. Let $f \in \mathrm{Jonq}_0(\mathbf{K})$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_2(\mathbf{K}(x))$ be a matrix representing f where $A,B,C,D \in \mathbf{K}[x]$. We introduce the function $\Delta := \frac{\mathrm{Tr}^2}{\det}$ which is well defined in PGL and is invariant by conjugation. This invariant Δ indicates the degree growth:

Lemma 2.13 ([CD12b] Theorem 3.3 [Xie15] Proposition 6.6) *The rational function $\Delta(f)$ is constant if and only if f is an elliptic element.*

Proof Let t_1,t_2 be the two eigenvalues of the matrix M which are elements of the algebraic closure of $\mathbf{K}(x)$. The invariant $\Delta(f)$ equals to $t_1/t_2 + t_2/t_1 + 2$. Since $\mathbf{K}$ is algebraically closed, $\Delta(f) \in \mathbf{K}$ if and only if $t_1/t_2 \in \mathbf{K}$. If $t_1 = t_2$, then by conjugating M to a triangular matrix we can write f in the form $(x,y) \dashrightarrow (x, y + a(x))$ with $a \in \mathbf{K}(x)$ and it follows that f is an elliptic element.

Suppose now that $t_1 \neq t_2$. Let $\zeta : C \to \mathbb{P}^1$ be the curve corresponding to the finite field extension $\mathbf{K}(x) \hookrightarrow \mathbf{K}(x)(t_1)$, here ζ is the identity map on $\mathbb{P}^1$ if $t_1,t_2 \in \mathbf{K}(x)$. The birational transformation f induces a birational transformation f_C on $C \times \mathbb{P}^1$ by base change. The induced map f_C is of the form $(x, (t_1/t_2)y)$ where t_1/t_2 is viewed as a function on C. The degree growth of f_C which is the same as f is linear if and only if t_1/t_2 is not a constant, i.e. if and only if $\Delta(f)$ is not a constant. $\square$

From now on we suppose that f is a Jonquières twist so that $\Delta(f) \notin \mathbf{K}$. We still denote by t_1,t_2 the two eigenvalues of M as in the above proof, we know that $t_1 \neq t_2$.

We first study the centralizer $\mathrm{Cent}_0(f)$ of f in $\mathrm{Jonq}_0(\mathbf{K}) = \mathrm{PGL}_2(\mathbf{K}(x))$. Let L be the finite extension of $\mathbf{K}(x)$ over which M is diagonalisable; it is $\mathbf{K}(x)$ itself or a quadratic extension of $\mathbf{K}(x)$, depending on whether or not t_1,t_2 are in $\mathbf{K}(x)$. The centralizer $\mathrm{Cent}_0^L(f)$ of f in $\mathrm{PGL}_2(L)$ is isomorphic to the multiplicative group L^*. So $\mathrm{Cent}_0(f)$, being contained in $\mathrm{Cent}_0^L(f)$ and containing all the iterates of f, must be a 1-dimensional torus over $\mathbf{K}(x)$. It is split if $L = \mathbf{K}(x)$, i.e. if $t_1,t_2 \in \mathbf{K}(x)$.

If $L = \mathbf{K}(x)$, then up to conjugation f can be written as $(x,y) \dashrightarrow (x, b(x)y)$ with $b \in \mathbf{K}(x)^*$ and $\mathrm{Cent}_0(f) = \{(x,y) \dashrightarrow (x, \gamma(x)y) | \gamma \in \mathbf{K}(x)^*\}$.

If L is a quadratic extension of $\mathbf{K}(x)$ and if $\mathrm{char}(\mathbf{K}) \neq 2$, we can put f in a simpler form and write $\mathrm{Cent}_0(f)$ explicitly as follows. We may assume that the matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ has entry $C = 1$, after conjugation by $\begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix}$. Once we have $C = 1$, a conjugation by $\begin{pmatrix} 2 & D-A \\ 0 & 2 \end{pmatrix}$ allows us to put M in the form $\begin{pmatrix} A & B \\ 1 & A \end{pmatrix}$ with $A,B \in \mathbf{K}[x]$. Therefore $\mathrm{Cent}_0(f)$ is $\{Id, (x,y) \dashrightarrow (x, \frac{C(x)y+B(x)}{y+C(x)}) | C \in \mathbf{K}(x)\}$ as the ($\mathbf{K}(x)$-points of the) later algebraic group is easily seen to commute with f. Note that B is not a square in $\mathbf{K}(x)$ because M is not diagonalisable over $\mathbf{K}(x)$ and that the transformation $f : (x,y) \dashrightarrow (x, \frac{A(x)y+B(x)}{y+A(x)})$ fixes pointwise the hyperelliptic curve defined by $y^2 = B(x)$.

Now we look at the whole centralizer of f. For $\eta \in \mathrm{PGL}_2(\mathbf{K})$ and $f \in \mathrm{Jonq}_0(\mathbf{K})$ represented by a matrix $\begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix}$, we denote by f_η the element of $\mathrm{Jonq}_0(\mathbf{K})$ represented by $\begin{pmatrix} A(\eta(x)) & B(\eta(x)) \\ C(\eta(x)) & D(\eta(x)) \end{pmatrix}$. Let $f \in \mathrm{Jonq}_0(\mathbf{K})$ be a Jonquières twist and $g : (x,y) \dashrightarrow (\eta(x), \frac{a(x)y+b(x)}{c(x)y+d(x)})$ be an element of $\mathrm{Jonq}(\mathbf{K})$. Writing down the commutation equation , we see that g commutes with f if and only if f is conjugate to f_η in $\mathrm{PGL}_2(\mathbf{K}(x))$ by the transformation represented by $\begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}$. We have thus $\Delta(f)(x) = \Delta(f_\eta)(x) = \Delta(f)(\eta(x))$. Recall that $\Delta(f) \in \mathbf{K}(x)$ is not in $\mathbf{K}$. As a consequence the group

$$\{\eta \in \mathrm{PGL}_2(\mathbf{K}), \Delta(f)(x) = \Delta(f)(\eta(x))\}$$

is a finite subgroup of $\mathrm{PGL}_2(\mathbf{K})$. We then obtain:

Theorem 2.14 ([CD12b]) *Let $f \in \mathrm{Jonq}_0(\mathbf{K})$ be a Jonquières twist preserving the rational fibration fibre by fibre. Let $\mathrm{Cent}(f)$ be the centralizer of f in $\mathrm{Cr}_2(\mathbf{K})$. Then $\mathrm{Cent}(f) \subset \mathrm{Jonq}(\mathbf{K})$ and $\mathrm{Cent}_0(f) = \mathrm{Cent}(f) \cap \mathrm{Jonq}_0(\mathbf{K})$ is a finite index normal subgroup of $\mathrm{Cent}(f)$. The group $\mathrm{Cent}_0(f)$ has a structure of a 1-dimensional torus over $\mathbf{K}(x)$. In particular $\mathrm{Cent}(f)$ is virtually abelian.*

Remark 2.15 In [CD12b], the authors give explicit description of the quotient $\mathrm{Cent}(f)/\mathrm{Cent}_0(f)$ when $\mathrm{char}(\mathbf{K}) = 0$.

Finite action on the base. If $f \in \mathrm{Jonq}(\mathbf{K})$ is a Jonquières twist which has a finite action on the base, then $f^k \in \mathrm{Jonq}_0(\mathbf{K})$ for some $k \in \mathbf{N}$. As $\mathrm{Cent}(f) \subset \mathrm{Cent}(f^k)$, we can use Theorem 2.14 to describe $\mathrm{Cent}(f)$:

Corollary 2.16 *If $f \in \mathrm{Jonq}(\mathbf{K})$ is a Jonquières twist which has a finite action on the base, then $\mathrm{Cent}(f)$ is virtually contained in a 1-dimensional torus over $\mathbf{K}(x)$. In particular $\mathrm{Cent}(f)$ is virtually abelian.*

We are contented with this coarse description of $\mathrm{Cent}(f)$ because this causes only a finite index problem as regards the embeddings of $\mathbf{Z}^2$ to $\mathrm{Cr}_2(\mathbf{K})$. We give an example to show how we expect $\mathrm{Cent}(f)$ to look like:

Example 2.17 If f is $(x,y) \dashrightarrow (a(x), R(x)y)$ where $R \in \mathbf{K}(x)$ and $a \in \mathrm{PGL}_2(\mathbf{K})$ is of order $k < +\infty$. Then all maps of the form $(x,y) \dashrightarrow (x, S(x)S(a(x)) \cdots S(a^{k-1}(x))y)$ with $S \in \mathbf{K}(x)$ commute with f.

3 Base-wandering Jonquières twists

We introduce some notations. For a Hirzebruch surface X, let us denote by π the projection of X onto $\mathbb{P}^1$, i.e. the rational fibration. When $X = \mathbb{P}^1 \times \mathbb{P}^1$, π is the projection onto the first factor. For $x \in \mathbb{P}^1$, we denote by F_x the fibre $\pi^{-1}(x)$. If f is a birational transformation of a Hirzebruch surface X which preserves the rational fibration, we denote by $\overline{f} \in \mathrm{PGL}_2(\mathbf{K})$ the induced action of f on the base $\mathbb{P}^1$ and we will consider f as an element of $\mathrm{Jonq}(\mathbf{K})$.

Assume now that f is a Jonquières twist such that $\overline{f} \in \mathrm{PGL}_2(\mathbf{K})$ if of infinite order, we will call it a *base-wandering Jonquières twist*. We have an exact sequence:

$$\{1\} \to \mathrm{Cent}_0(f) \to \mathrm{Cent}(f) \to \mathrm{Cent}_b(f) \to \{1\} \tag{2}$$

where $\mathrm{Cent}_0(f) = \mathrm{Cent}(f) \cap \mathrm{Jonq}_0(\mathbf{K})$ and $\mathrm{Cent}_b(f) \subset \mathrm{Cent}(\overline{f}) \subset \mathrm{PGL}_2(\mathbf{K})$. The action $\overline{f}$ on the base is conjugate to $x \mapsto \alpha x$ with $\alpha \in \mathbf{K}^*$ of infinite order or to $x \mapsto x+1$. The later case is only possible if $\mathrm{char}(\mathbf{K}) = 0$. Thus $\mathrm{Cent}_b(f)$ is a subgroup of $\{x \mapsto \gamma x, \gamma \in \mathbf{K}^*\}$ or of $\{x \mapsto x+\gamma, \gamma \in \mathbf{K}\}$. In both cases $\mathrm{Cent}_b(f)$ is abelian. We first remark:

Lemma 3.1 *All elements of* $\mathrm{Cent}_0(f)$ *are elliptic.*

Proof By Theorem 2.14, a Jonquières twist in $\mathrm{Jonq}_0(\mathbf{K})$ can not have a base-wandering Jonquières twist in its centralizer. □

The rest of the article will essentially be occupied by the proof of the following theorem:

Theorem 3.2 *Let* $f \in \mathrm{Jonq}(\mathbf{K})$ *be a base-wandering Jonquières twist. The exact sequence*

$$\{1\} \to \mathrm{Cent}_0(f) \to \mathrm{Cent}(f) \to \mathrm{Cent}_b(f) \to \{1\}$$

satisfies

- $\mathrm{Cent}_0(f) = \mathrm{Cent}(f) \bigcap \mathrm{Jonq}_0(\mathbf{K})$*, if not trivial, is* $\{(x,y) \mapsto (x,ty), t \in \mathbf{K}^*\}$*,* $\{(x,y) \mapsto (x,y+t), t \in \mathbf{K}\}$*,* $\langle (x,y) \mapsto (x,-y) \rangle$ *or* $\langle$*a Jonquières involution*$\rangle$*;*
- $\mathrm{Cent}_b(f) \subset \mathrm{PGL}_2(\mathbf{K})$ *is isomorphic to the product of a finite cyclic group with* $\mathbf{Z}$*. The infinite cyclic subgroup generated by* $\overline{f}$ *has finite index in* $\mathrm{Cent}_b(f)$*.*

Proof (of Theorem 3.2) The theorem is a consequence of Proposition 3.15, Corollary 3.19 and Proposition 3.31. □

Corollary 3.3 *The centralizer of a base-wandering Jonquières twist is virtually abelian.*

Proof This results directly from the fact that $\mathrm{Cent}_b(f)$ is virtually the cyclic group generated by $\overline{f}$. □

Remark 3.4 Theorem 3.2 is optimal in the sense that $\mathrm{Cent}_b(f)$ can be $\mathbf{Z}$ (Remark 3.5) or a product of $\mathbf{Z}$ with a non trivial finite cyclic group (Example 3.30) and $\mathrm{Cent}_0(f)$ can be trivial, isomorphic to $\mathbf{K}$, $\mathbf{K}^*$ or $\mathbf{Z}/2\mathbf{Z}$ (Section 3.2).

Remark 3.5 A general base-wandering Jonquières twist can not be written as $(\eta(x), yR(x^k))$ or $(\eta(x), y+R(x))$. So the centralizer of a general Jonquières twist f differs from the infinite cyclic group $\langle f \rangle$ only by some finite groups. For example, for a generic choice of $\alpha, \beta \in \mathbf{K}^*$, the centralizer of $f_{\alpha,\beta} : (x,y) \dashrightarrow (\alpha x, \frac{\beta y + x}{y+1})$ is $\langle f_{\alpha\beta} \rangle$, this is showed by J.Déserti in [D0́8].

3.1 Algebraically stable maps

If f is a birational transformation of a smooth algebraic surface X over $\mathbf{K}$, we denote by $\mathrm{Ind}(f)$ the set of indeterminacy points of f. We say that f is *algebraically stable* if there is no curve V on X such that the strict transform $f^k(V) \subset \mathrm{Ind}(f)$ for some integer $k \geq 0$. There always exists a birational morphism $\hat{X} \to X$ which lifts f to an algebraically stable birational transformation of $\hat{X}$ ([DF01] Theorem 0.1). The following theorem says that for $f \in \mathrm{Jonq}(\mathbf{K})$, we can get a more precise algebraically stable model:

Theorem 3.6 *Let f be a birational transformation of a ruled surface X that preserves the rational fibration. Then there is a rational ruled surface $\hat{X}$ and a birational map $\varphi : X \dashrightarrow \hat{X}$ such that*

- *the only singular fibres of $\hat{X}$ are of the form $D_0 + D_1$ where D_0, D_1 are (-1)-curves, i.e. $\hat{X}$ is a conic bundle;*
- *$f_{\hat{X}} = \varphi \circ f \circ \varphi^{-1}$ is an algebraically stable birational transformation of $\hat{X}$ and it preserves the rational fibration of $\hat{X}$ which is induced by that of X;*
- *$f_{\hat{X}}$ sends singular fibres isomorphically to singular fibres and all indeterminacy points of $f_{\hat{X}}$ and its iterates are located on regular fibres.*
- *φ is a sequence of elementary transformations and blow-ups.*

Let $z \in X$ be an indeterminacy point of f. Let $X \xleftarrow{u} Y \xrightarrow{v} X$ be a minimal resolution of the indeterminacy point z, i.e. u, v are birational maps which are regular around the fibre over $\pi(z)$, u^{-1} is a series of n blow-ups at z or at its infinitely near points and n is minimal among possible integers.

Lemma 3.7 *The total transform by u^{-1} in Y of $F_{\pi(z)}$, the fibre containing z, is a chain of $(n+1)$ rational curves $C_0 + C_1 + \cdots + C_n$: C_0 is the strict transform of $F_{\pi(z)}$, $C_0^2 = C_n^2 = -1$, $C_i^2 = -2$ for $0 < i < n$ and $C_i \cdot C_{i+1} = 1$ for $0 \leq i < n$.*

Proof Let us write $u : Y \to X$ as $Y = Y_n \xrightarrow{u_n} Y_{n-1} \cdots \xrightarrow{u_2} Y_1 \xrightarrow{u_1} Y_0 = X$ where each u_i is a single contraction of a (-1)-curve and C_i is (the strict transform) of the contracted (-1)-curve. By an abuse of notation, we will use C_i to denote all strict transforms of the (-1)-curve contracted by u_i. The connectedness of the fibres and the preservation of the fibration imply that for each i, the map $f \circ u_1 \circ \cdots \circ u_i$ has at most one indeterminacy point on a fibre. To prove the lemma, it suffices to show that the indeterminacy point of $f \circ u_1 \circ \cdots \circ u_i$ which by construction lies in C_i is not the intersection point of C_i with C_{i-1}.

Suppose by contradiction that C_{i+1} is obtained by blowing up the intersection point of C_i with C_{i-1}. Then for $j > i$, the auto-intersection of C_i on X_j is less than or equal to -2. Let us write $v : Y \to X$ as $Y = Y_n \xrightarrow{v_n} Y_{n-1} \cdots \xrightarrow{v_2} Y_1 \xrightarrow{v_1} Y_0 = X$ where each v_i is a single contraction of a (-1)-curve. Since C_i is contracted by v, there must exist an integer k such that $v_{k+1} \circ \cdots \circ v_n(C_i)$ is the (-1)-curve on Y_k contracted by v_k. This is possible only if the $C_j, j > i$ are all contracted by $v_k \circ \cdots \circ v_n$. But by the minimality of the integer n, C_n can not be contracted by v. $\square$

Proof (of Theorem 3.6) Our proof is inspired by the proof of Theorem 0.1 of [DF01]. Let $p_1, \cdots, p_k \in X$ be the indeterminacy points of f. By Lemma 3.7, for $1 \leq i \leq k$ the minimal resolution of f at p_i writes as

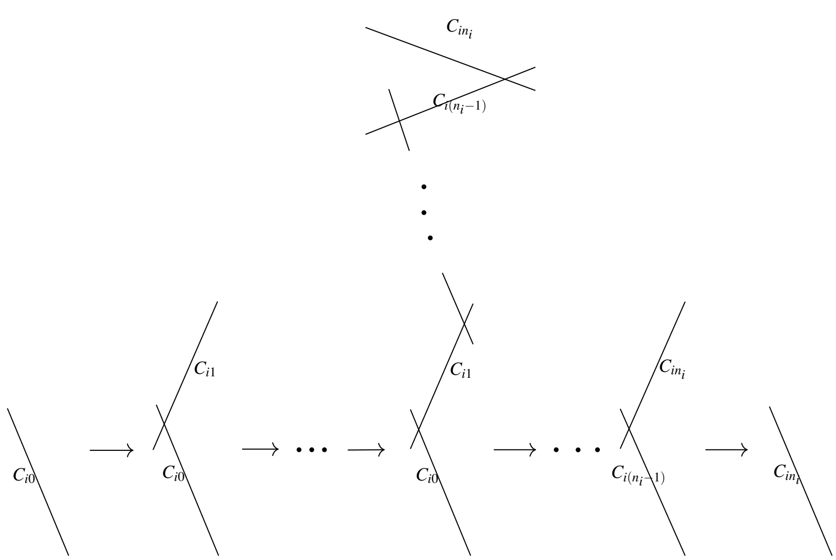
$$X = X_{i0} \xleftarrow{u_{i1}} X_{i1} \xleftarrow{u_{i2}} \cdots \xleftarrow{u_{in_i}} X_{in_i} = Y_{in_i} \xrightarrow{v_{in_i}} \cdots \xrightarrow{v_{i2}} Y_{i1} \xrightarrow{v_{i1}} Y_{i0} = X$$

where $u_{i1}, \cdots, u_{in_i}, v_{i1}, \cdots, v_{in_i}$ are single contractions of (-1)-curves and X_{in_i} has one singular fibre which is a chain of rational curves $C_{i0} + \cdots + C_{in_i}$. Let us write the global minimal resolution of indeterminacy of f by keeping in mind the rational fibration:

$$\begin{array}{ccccccccccccc}
X = X_0 & \xrightarrow{f_0} & X_1 & \xrightarrow{f_1} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \cdots & \xrightarrow{f_{2n-2}} & X_{2n-1} & \xrightarrow{f_{2n-1}} & X_{2n} = X \\
\downarrow \pi & & \downarrow \pi & & & & \downarrow \pi & & & & \downarrow \pi & & \downarrow \pi \\
\mathbb{P}^1 & \xrightarrow{\overline{f_0}} & \mathbb{P}^1 & \xrightarrow{\overline{f_1}} & \cdots & \xrightarrow{\overline{f_{n-1}}} & \mathbb{P}^1 & \xrightarrow{\overline{f_n}} & \cdots & \xrightarrow{\overline{f_{2n-2}}} & \mathbb{P}^1 & \xrightarrow{\overline{f_{2n-1}}} & \mathbb{P}^1
\end{array}$$

where $n = n_1 + \cdots + n_k$ and

- $f_0, \cdots, f_{n-1}$ are blow-ups which correspond to the inverses of $u_{11}, \cdots, u_{1n_1}, \cdots, u_{k1}, \cdots, u_{kn_k}$;
- $f_n, \cdots, f_{2n-1}$ are blow-downs which correspond to $v_{11}, \cdots, v_{1n_1}, \cdots, v_{k1}, \cdots, v_{kn_k}$;
- X_n has k singular fibres which are chains of rational curves $C_{i0} + \cdots + C_{in_i}, 1 \leq i \leq k$;
- the abusive notation π is self-explaining and we will also denote by C_{il} its strict transforms (if it remains a curve) on the surfaces X_j. On $X_0 = X_{2n}$, it is possible that $C_{i'0} = C_{in_i}$ for $1 \leq i, i' \leq k$.



For any $j \in \mathbf{N}$, we let $X_j = X_{j \mod 2n}$ and $f_j = f_{j \mod 2n}$. If f_j blows up a point $r_j \in X_j$, then we denote by V_{j+1} the exceptional curve on X_{j+1}. If f_j contracts a curve $W_j \subset X_j$ then we denote by s_{j+1} the point $f_j(W_j) \in X_{j+1}$. For each V_j (resp. W_j), there is an i such that V_j (resp. W_j) is among $C_{i0}, \cdots, C_{in_i}$. Suppose that f is not algebraically stable on H. Then there exist integers $1 \leq M < N$ such that f_M contracts W_M and

$$f_{N-1} \circ \cdots \circ f_M(W_M) = r_N \in \mathrm{Ind}(f_N).$$

We can assume that $n \leq N \leq 2n-1$ and the length $(N-M)$ is minimal. Observe first that the minimality of the length implies for all $M \leq j < N-1$, the point $t_{j+1} := f_j \circ \cdots \circ f_M(W_M) = f_j \circ \cdots \circ f_{M+1}(s_{M+1})$ is neither an indeterminacy point nor a point on a curve contracted by f_{j+1}. Secondly we assert that for all $M \leq j < N-1$, t_{j+1} is not on the singular fibres of X_{j+1}. Indeed if some t_{j+1} was on a singular fibre of X_{j+1}, then the sequence of points $t_{j+1}, t_{j+2}, \cdots$ would meet a contracted curve before meeting the first indeterminacy point r_N (look at the picture), which contradicts our first observation. The second observation further implies that for $M \leq j < N-1$ such that $j+2n < N-1$, t_{j+1}, t_{j+2n+1} are not on the same fibre of $X_{j+1} = X_{j+2n+1}$ because otherwise there would exist $j < j' < j+2n+1$ such that $j' = M \mod 2n$ and $t_{j'}$ would be on the singular fibre containing W_M.

Since f_{N-1} maps isomorphically the fibre of X_{N-1} containing t_{N-1} (which is regular by the above observation) to the fibre of X_N containing r_N, the fibre containing r_N is just one rational curve. As f_N is a blow-up, the fibre of X_{N+1} containing V_{N+1} is the union of two (-1)-curves, let us say, C_{k0} and $C_{k1} = V_{N+1}$. Then the fibre of X_N containing r_N is just C_{k0}. Similarly the singular fibre of X_M containing W_M is $C_{mn_m} + C_{m(n_m-1)}$ for some $1 \leq m \leq k$.

First case. Suppose that $m = k$ and $n_k = 1$. Let $a \in \mathbf{N}$ be the minimal integer such that $M + 2an > N$. Then for $N < j \leq M + 2an$, the surface X_j has a singular fibre $C_{k0} + C_{k1}$ and the maps $f_N, \cdots, f_{M+2an-1}$ are all regular on $C_{k0} + C_{k1}$. Now we blow-up $t_{M+1}, \cdots, t_{N-1}, r_N$. For $j_1 = j_2 \mod 2n$, we showed that t_{j_1}, t_{j_2} are not on the same fibre of $X_{j_1} = X_{j_2}$. This means that these blow-ups only give rise to singular fibres which are unions of two (-1)-curves. We denote by $\hat{X}_j$ the modified surfaces, and $\hat{f}_j$ the induced maps. Then every $\hat{X}_j$ has singular fibres of the form $C_{k0} + C_{k1}$ and every $\hat{f}_j$ is regular around these singular fibres. Let $\hat{f} = \hat{f}_{2n-1} \circ \cdots \circ \hat{f}_0$. The number of indeterminacy points of $\hat{f}$ (it was k for f) has decreased by one. Note that $\hat{f}$ exchanges the two components C_{k0} and C_{k1}. This fact will be used in the proof of Corollary 3.8.

Second case. Suppose that $m = k$ and $n_k > 1$ or simply $m \neq k$. We blow-up r_N and contract the strict transform of the initial fibre containing r_N which is C_{k0}, obtaining a new surface $\hat{X}_N$ whose corresponding fibre is now the single rational curve C_{k1}. We perform elementary transformations at $t_{N-1}, \cdots, t_{M+1}$, i.e. we blow-up X_j at t_j and contract the strict transform of the initial fibre, replacing X_j with $\hat{X}_j$. This process has no ambiguity: if $j_1 = j_2 \mod 2n$, we showed that t_{j_1}, t_{j_2} are not on the same fibre of $X_{j_1} = X_{j_2}$, so the corresponding elementary transformations do not interfere with each other. Let us denote by $\hat{f}_M, \cdots, \hat{f}_N$ the maps induced by $f_M, \cdots, f_N$.

We now analyse the effects of $\hat{f}_M, \cdots, \hat{f}_N$. First look at f_N, it lifts to a regular isomorphism after blowing up r_N. Thus $\hat{f}_N$ is the blow-up at the point e_N of $\hat{X}_N$ to which C_{k0} is contracted. After this step, the map going from X_{N-1} to $\hat{X}_N$ induced by f_{N-1} is as following: it contracts the fibre containing t_{N-1} to e_N and blows up t_{N-1}. Then we make elementary transformations at $t_{N-1},\cdots,t_{M+1}$ in turn. The maps $\hat{f}_{N-1},\cdots,\hat{f}_{M+1}$ are all regular on the modified fibres, thus they are still single blow-ups or single blow-downs. The behaviour of $\hat{f}_M$ differs from the previous ones: it does not contract $C_{m(n_m-1)}$ any more, but contracts C_{mn_m}.

The hypothesis $m \neq k$ (or $m = k$, $n_k > 1$) forbids $C_{k0} \subset X_{N+1}$ to go back into the fibre of $X_{M+2na} = X_M$ containing W_M without being contracted. More precisely this implies the existence of $N' > N$ such that

- $X_{N+1},\cdots,X_{N'}$ all contain C_{k0} and C_{k1};
- $f_{N+1},\cdots,f_{N'-1}$ are regular on C_{k0} and $f_{N'}$ contracts C_{k0};
- if $a \in \mathbf{N}$ is the minimal integer such that $M+2na > N$, then $N' < M+2na$.

On the surfaces $X_{N+1},\cdots,X_{N'}$, C_{k0} is always a (-1)-curve, we contract all these C_{k0} and obtain new surfaces $\hat{X}_{N+1},\cdots,\hat{X}_{N'}$. The second and the third property listed above mean that the new induced maps $\hat{f}_N,\cdots,\hat{f}_{N'}$ are all single blow-ups, single blow-downs or simply isomorphisms.

In summary we get a commutative diagram:

$$\begin{array}{ccccccccccccc}
\hat{X}_0 & \xrightarrow{\hat{f}_0} & \hat{X}_1 & \xrightarrow{\hat{f}_1} & \cdots & \xrightarrow{\hat{f}_{n-1}} & \hat{X}_n & \xrightarrow{\hat{f}_n} & \cdots & \xrightarrow{\hat{f}_{2n-2}} & \hat{X}_{2n-1} & \xrightarrow{\hat{f}_{2n-1}} & \hat{X}_{2n} = \hat{X}_0 \\
\downarrow & & \downarrow & & & & \downarrow & & & & \downarrow & & \downarrow \\
X_0 & \xrightarrow{f_0} & X_1 & \xrightarrow{f_1} & \cdots & \xrightarrow{f_{n-1}} & X_n & \xrightarrow{f_n} & \cdots & \xrightarrow{f_{2n-2}} & X_{2n-1} & \xrightarrow{f_{2n-1}} & X_{2n} = X_0
\end{array}$$

where the vertical arrows are composition of elementary transformations and blow-ups. Let us remark that:

- the first vertical arrow $\hat{X}_0 \dashrightarrow X_0$ is a composition of elementary transformations.
- the blow-ups or the contractions of the $\hat{f}_j$ only concern the k singular fibres and the exceptional curves are always among $C_{10},\cdots,C_{1n_1},\cdots,C_{k0},\cdots,C_{kn_k}$;
- there is no more C_{k0}. We then do a renumbering: $C_{k1},\cdots,C_{kn_k}$ become $C_{k0},\cdots,C_{k(n_k-1)}$.

Let $\hat{f} = \hat{f}_{2n-1} \circ \cdots \circ \hat{f}_0$. We repeat the above process. Either we are in the first case and k decreases, or we are in the second case and the total number of $C_{10},\cdots,C_{1n_1},\cdots,C_{k0},\cdots,C_{kn_k}$ decreases. As a consequence, after a finite number of times, either we get an algebraically stable map $\hat{f}$, or we will get rid of all the $C_{10},\cdots,C_{1n_1},\cdots,C_{k0},\cdots,C_{kn_k}$. In the later case $\hat{f}$ is a regular automorphism, thus automatically algebraically stable. $\square$

Theorem 3.6 also gives a geometric complement to the study of elements of finite order of $\mathrm{Jonq}(\mathbf{K})$ in [Bla11] Section 3. In particular the proof of Theorem 3.6 implies the following corollary (which is already known, see for example [Bla11]), one special case of which will be used in the next section:

Corollary 3.8 *Let $f \in \mathrm{Jonq}(\mathbf{K})$ be an elliptic element. If f is not conjugate to an automorphism of a Hirzebruch surface, then it is a conjugate to an automorphism of a conic bundle and the order of f is $2k$ for some $k \in \mathbf{N}^*$. Moreover f^k is in $\mathrm{Jonq}_0(\mathbf{K})$ and exchanges the two components of some singular fibres of the conic bundle.*

Proof We see by Theorem 2.4 that an elliptic element of infinite order is always conjugate to an automorphism of Hirzebruch surface. Hence our hypothesis implies immediately that f is of finite order. We can assume that f is an algebraically stable map on a conic bundle X which satisfies the conditions of Theorem 3.6. We claim that f is an automorphism of X. Suppose by contradiction that p is an indeterminacy point of f. It must lie on a regular fibre F of X. The fact that f is of finite order and the algebraic stability of f imply that f^{-1} has an indeterminacy point on F different from p. But then f can not be of finite order, contradiction.

Since by hypothesis X is not a Hirzebruch surface, it must have some singular fibres. By the proof of Theorem 3.6 (see the *First case* in the proof), for each singular fibre there exists an iterate of f which exchanges the two components of that fibre. Since there are finitely many singular fibres, we can find an integer $k > 0$ such that f^k is in $\mathrm{Jonq}_0(\mathbf{K})$ and exchanges the two components of at least one singular fibre. If we consider f^k as an element of $\mathrm{PGL}_2(\mathbf{K}(x))$, it is not diagonalizable over $\mathbf{K}(x)$. As we have seen in Section 2.4, the map f^k, being non diagonalizable, fixes pointwise a hyperelliptic curve whose projection onto $\mathbb{P}^1$ is induced by the rational fibration. The map f^{2k} does not exchange the components of the singular fibres, so it is conjugate to an automorphism of a Hirzebruch surface and is diagonalizable over $\mathbf{K}(x)$. A diagonalizable map does not fix any hyperelliptic curve like this unless the map is trivial. Hence $f^{2k} = \mathrm{Id}$. □

See [Bla11] Section 3, especially Proposition 3.3 and Lemma 3.9, for more information on such elliptic elements of finite order; see also [DI09]. We will use a special case of the above corollary:

Corollary 3.9 *Let $f \in \mathrm{Jonq}_0(\mathbf{K})$ be an elliptic element which is not conjugate to an automorphism of a Hirzebruch surface. Then f is of order 2 and is conjugate to an automorphism of a conic bundle on which it fixes pointwise a hyperelliptic curve whose projection onto the base $\mathbb{P}^1$ is a ramified double cover. In some affine chart f writes as $(x,y) \dashrightarrow (x, \frac{a(x)}{y})$ with $a \in \mathbf{K}[x]$. The hyperelliptic curve is given by the equation $y^2 = a(x)$.*

Such involutions are well known and are called *Jonquières involutions*, see [BB00].

Remark 3.10 An element of the form $(x,y) \dashrightarrow (\eta(x), yR(x))$ or $(x,y) \dashrightarrow (\eta(x), y + R(x))$ with $\eta \in \mathrm{PGL}_2(\mathbf{K})$ and $R \in \mathbf{K}(x)$ is never a Jonquières twist. Thus by Theorem 2.7, a Jonquières twist never commutes with an elliptic element of infinite order.

We will need an abelian elliptic group version of Theorem 3.6:

Corollary 3.11 *Let $G \subset \mathrm{Jonq}(\mathbf{K})$ be a finitely generated abelian elliptic subgroup without Jonquières involutions. We can conjugate G to a group of automorphisms of a Hirzebruch surface. The conjugation is a sequence of elementary transformations.*

Proof Let $f_1, \cdots, f_d \in G$ be a finite set of generators of G. We apply Theorem 3.6 to f_1, then to f_2, etc. Remark that by the proof of Theorem 3.6, the elementary transformations of the conjugation are made at the indeterminacy points of the f_i. However G

is an abelian group, so that if p is an indeterminacy point of f_i and g is another element of G, then either g fixes p or p is an indeterminacy point of g too. Therefore after applying Theorem 3.6 to f_{i+1}, the previous ones $f_1, \cdots, f_i$ remain automorphisms. $\square$

3.2 The group $\mathrm{Cent}_0(f)$

Let f be a base-wandering Jonquières twist. In [CD12b], it is proved by explicit calculations, in the case where $\mathbf{K} = \mathbf{C}$, that $\mathrm{Cent}_0(f)$ is isomorphic to $\mathbf{C}^*$, $\mathbf{C}^* \rtimes \mathbf{Z}/2\mathbf{Z}$, $\mathbf{C}$ or a finite group (this is not optimal). Their arguments do not work directly when $\mathrm{char}(\mathbf{K}) > 0$. With a more precise description of elements of $\mathrm{Jonq}_0(\mathbf{K})$, we simplify their arguments and improve their results.

Let $g \in \mathrm{Cent}_0(f)$ be non trivial. Then either g is conjugate to an automorphism of a Hirzebruch surface or g is a Jonquières involution as in Corollary 3.9. In the first case, by proposition 2.6, we can write g as $(x,y) \mapsto (x, \beta y)$ or $(x,y) \mapsto (x, y+1)$.

Lemma 3.12 *Suppose that there exists a non trivial $g \in \mathrm{Cent}_0(f)$ that writes as $(x,y) \mapsto (x, \beta y)$ with $\beta \in \mathbf{K}^*$. Either f is of the form $(a(x), R(x)y^{-1})$ and $\mathrm{Cent}_0(f)$ is an order two group generated by the involution $(x,y) \mapsto (x,-y)$, or f is of the form $(a(x), R(x)y)$ and $\mathrm{Cent}_0(f)$ is $\{(x,y) \mapsto (x, \gamma y), \gamma \in \mathbf{K}^*\}$.*

Proof The map g preserves $\{y = 0\}$ and $\{y = \infty\}$ and these two curves are the only g-invariant sections. Thus f permutes these two sections and is necessarily of the form $(x,y) \dashrightarrow (a(x), R(x)y^{\pm 1})$ where $R \in \mathbf{K}(x)$ and $a \in \mathrm{PGL}_2(\mathbf{K})$ is of infinite order. If f is $(a(x), R(x)y^{-1})$, then $\beta = -1$. For the discussion which follows, it is not harmfull to replace f by f^2 so that we can assume f is $(a(x), R(x)y)$.

The only f-invariant sections are $\{y = 0\}$ and $\{y = \infty\}$. Indeed an invariant section s satisfies

$$s(a^n(x)) = R(x) \cdots R(a^{n-1}(x)) s(x) \quad \forall n \in \mathbf{N}.$$

If s was not $\{y = 0\}$ nor $\{y = \infty\}$, then the two sides of the equations are rational fractions and by comparing the degrees (of numerators and denominators) we get a contradiction because R is not constant. Thus, an element of $\mathrm{Cent}_0(f)$ permutes the two f-invariant sections and is of the form $(x, A(x)y)$ or $(x, \frac{A(x)}{y})$ with $A \in \mathbf{K}(x)$. In the first case the commutation relation implies $A(a(x)) = A(x)$ which further implies that A is a constant. In the second case the commutation relation gives $A(a(x))^{-1} R(x)^2 A(x) = 1$ which further implies that $(a(x), R(x)^2 y)$ is conjugate by $(x, A(x)y)$ to an elliptic element $(a(x), y)$. This is not possible because the map $f' : (x,y) \dashrightarrow (a(x), R(x)^2 y)$ is a Jonquières twist. Indeed the iterates f^n, f'^n are respectively

$$(a^n(x), R(x) \cdots R(a^{n-1}(x))y) \quad \text{and} \quad (a^n(x), (R(x) \cdots R(a^{n-1}(x)))^2 y)$$

and they have the same degree growth.

Reciprocally all elements of the form $(x,y) \mapsto (x, \beta y)$ with $\beta \in \mathbf{K}^*$ commute with $f : (x,y) \dashrightarrow (a(x), R(x)y)$ and we have already observed that $(x,y) \mapsto (x,-y)$ is the only non trivial element of $\mathrm{Jonq}_0(\mathbf{K})$ which commutes with $(a(x), R(x)y^{-1})$. $\square$

Lemma 3.13 *Suppose that there exists a non trivial $g \in \mathrm{Cent}_0(f)$ that writes as $(x,y) \mapsto (x, y+1)$. Then f is of the form $(a(x), y + S(x))$ with $S \in \mathbf{K}(x)$ and $\mathrm{Cent}_0(f)$ is $\{(x, y + \gamma), \gamma \in \mathbf{K}\}$.*

Proof The section $\{y=\infty\}$ is the only g-invariant section. Thus f preserves this section and is of the form $(x,y) \dashrightarrow (a(x), R(x)y+S(x))$ where $R,S \in \mathbf{K}(x)$ and $a \in \mathrm{PGL}_2(\mathbf{K})$ is of infinite order. Writing down the relation $f\circ g = g\circ f$, we see that $R=1$. Thus f is $(a(x), y+S(x))$ where S belongs to $\mathbf{K}(x)$ but not to $\mathbf{K}[x]$ since f is a Jonquières twist. The only f-invariant section is $\{y=\infty\}$. Indeed an invariant section s satisfies

$$s(a^n(x)) = s(x)+S(x)+\cdots+S(a^{n-1}(x)) \quad \forall n \in \mathbf{N}.$$

If s was not $\{y=\infty\}$, then the two sides of the equations are rational fractions. The degree of the right-hand side grows linearly in n while the degree of the left-hand side does not depend on n, contradiction. Thus, an element of $\mathrm{Cent}_0(f)$ fixes $\{y=\infty\}$ and is of the form $(x, A(x)y+B(x))$ with $A,B \in \mathbf{K}(x)$. Writing down the commutation relation, we get

$$A(x)y+B(x)+S(x) = A(a(x))y+A(a(x))S(x)+B(a(x)).$$

The fact that a is of infinite order implies that A is a constant. Then the equation is reduced to

$$B(x)+(1-A)S(x)-B(a(x)) = 0.$$

If $A\neq 1$, then $f:(x,y)\dashrightarrow (a(x),y+S(x))$ would be conjugate by $(x, y+\frac{B(x)}{1-A})$ to the elliptic elment $(a(x),y)$. Therefore $A=1$ and B is a constant. Reciprocally we see that all elements of the form $(x,y)\mapsto(x,y+\beta)$ with $\beta\in\mathbf{K}$ commute with $f:(x,y)\dashrightarrow (a(x),y+S(x))$. $\square$

Lemma 3.14 *Assume that no non-trivial element of* $\mathrm{Cent}_0(f)$ *is conjugate to an automorphism of a Hirzebruch surface and that* $\mathrm{Cent}_0(f)$ *has a non-trivial element g. Then g is a Jonquières involution and is the only non-trivial element of* $\mathrm{Cent}_0(f)$.

Proof By Lemma 3.1, g is an elliptic element. By Corollary 3.9, g acts on a conic bundle X and fixes pointwise a hyperelliptic curve C. The map f induces an action on C, equivariant with respect to the ramified double cover. The action of f on C is infinite, this is possible only if the action of f on the base is up to conjugation $x\mapsto\alpha x$ and if C is a rational curve whose projection on the base $\mathbb{P}^1$ is ramified over $x=0, x=\infty$. Then the only singular fibres of X are over $x=0, x=\infty$. If f had an indeterminacy point on these two fibres, then it would be a fixed point of g because g commutes with f. But the only fixed point of g on a singular fibre is the intersection point of the two components, which can not be an indeterminacy point by Lemma 3.7. Therefore the Jonquières twist f must have an indeterminacy point over a point whose orbit in the base is infinite. This implies that the indeterminacy points of all the iterates of f form an infinite set. As g commutes with all the iterates of f, it fixes an infinite number of these indeterminacy points. Thus, the hyperelliptic curve C associated to g is the Zariski closure of these indeterminacy points and is uniquely determined by f. However C determines g too by Corollary 3.9 (see [Bla11] for more general results). Therefore g is uniquely determined by f and is the only non trivial element of $\mathrm{Cent}_0(f)$. $\square$

Putting together the three previous lemmas, we obtain the following improvement of [CD12b]:

Proposition 3.15 *Let f be a base-wandering Jonquières twist. If* $\mathrm{Cent}_0(f)$ *is not trivial, then it is* $\{(x,y)\mapsto(x,ty), t\in\mathbf{K}^*\}$, $\{(x,y)\mapsto(x,y+t), t\in\mathbf{K}\}$, $\langle(x,y)\mapsto(x,-y)\rangle$ *or* $\langle$*a Jonquières involution*$\rangle$.

3.3 Persistent indeterminacy points

3.3.1 general facts

Let f be a birational transformation of a surface X. An indeterminacy point $x\in X$ of f will be called *persistent* if 1) for every $i>0$, f^{-i} is regular at x; and 2) there are infinitely many curves contracted onto x by the iterates f^{-n}, $n\in\mathbf{N}$. This notion of persistence and the following idea appeared first in a non published version of [Can11], and it is also applied to some particular examples in [D08́].

Proposition 3.16 *Let f be an algebraically stable birational transformation of a surface X. Suppose that there exists at least one persistent indeterminacy point with an infinite backward orbit. Let n denote the number of such indeterminacy points. Then the centralizer* $\mathrm{Cent}(f)$ *of f admits a morphism* $\varphi:\mathrm{Cent}(f)\to\mathscr{S}_n$ *to the symmetric group of order n satisfying the following property: for any* $g\in\mathrm{Ker}(\varphi)$*, there exists $l\in\mathbf{Z}$ such that $g\circ f^l$ preserves fibre by fibre a pencil of rational curves.*

Proof The algebraic stability of f will be used throughout the proof, we will not recall it each time. Denote by $p_1,\cdots,p_n$ the persistent indeterminacy points of f. Let g be a birational transformation of X which commute with f. Fix an index $1\leq n_0\leq n$. Since $\{f^{-i}(p_{n_0}), i>0\}$ is infinite, there exists $k_0>0$ such that g is regular at $f^{-k}(p_{n_0})$ for all $k\geq k_0$. For infinitely many $j>0$, f^{-j} contracts a curve onto p_1, denote these curves by $C^j_{n_0}$. There exists $k_1>0$ such that g does not contract $C^k_{n_0}$ for all $k\geq k_1$. We deduce, from the above observations and the fact that f and g commute, that for $k\geq k_0$ the point $g(f^{-k}(p_{n_0}))$ is an indeterminacy point of some f^m with $0<m\leq k_0+k_1$. Then there exists $0\leq m_0<m$ such that

- for $0\leq i\leq m_0$, f^i is regular at $g(f^{-k}(p_{n_0}))$;
- $f^{m_0}(g(f^{-k}(p_{n_0})))=g(f^{m_0-k}(p_{n_0}))$ is an indeterminacy point of f.

By looking at $g(f^{-k}(p_{n_0}))$ and $C^{k'}_{n_0}$ for infinitely many k,k', we see that the above indeterminacy point does not depend on k and is persistent with an infinite backward orbit. So it is $p_{\sigma_g(n_0)}$ for some $1\leq\sigma_g(n_0)\leq n$. This gives us a well defined map $\sigma_g:\{1,\cdots,n\}\to\{1,\cdots,n\}$.

Now let g,h be two elements of $\mathrm{Cent}(f)$. Then by considering a sufficiently large k for which g is regular at $f^{-k}(p_{n_0})$ and h is regular at $g(f^{-k}(p_{n_0}))$, we see that $\sigma_h\circ\sigma_g=\sigma_{h\circ g}$. By taking $h=g^{-1}$ we see that σ_g is bijective. We have then a group homomorphism φ from $\mathrm{Cent}(f)$ to the symmetric group $\mathscr{S}_n$ which sends g to σ_g.

Assume that n_0 is a fixed point of σ_g, this holds in particular when $g\in\mathrm{Ker}(\varphi)$. We keep the previous notations. Since $g(f^{-k}(p_{n_0}))$ is an indeterminacy point of f^m whose forward orbit meets p_{n_0}, for an appropriate choice of $l\leq k$ we have

$$g\circ f^l(f^{-k}(p_{n_0}))=f^{-k}(p_{n_0})$$

for all $k \geq k_0$. This implies further

$$g \circ f^l(C_{n_0}^{k'}) = C_{n_0}^{k'}$$

for all sufficiently large k'. We conclude by Lemma 3.17 below. $\square$

The proof of the following lemma in [Can10] is written over $\mathbf{C}$ for rational self-maps. It is observed in [Xie15] that the same proof works in all characteristics for birational transformations.

Lemma 3.17 *A birational transformation of a smooth algebraic surface which preserves infinitely many curves preserves each member of a pencil of curves.*

3.3.2 persistent indeterminacy points for Jonquières twists

We examine the notion of persistence in the Jonquières group and give a complement to Theorem 3.6:

Proposition 3.18 *Let f be a Jonquières twist acting algebraically stably on a conic bundle X as in the statement of Theorem 3.6. Then an indeterminacy point p of f is persistent if and only if the orbit of $\pi(p) \in \mathbb{P}^1$ under $\overline{f}$ is infinite. And in that case, every $f^{-i}, i \in \mathbf{N}^*$ contracts a curve onto p.*

Proof If $\pi(p)$ has a finite orbit then p certainly can not be persistent. Let us assume that the orbit of $\pi(p)$ is infinite. Then $\overline{f}$ is conjugate to $x \mapsto \alpha X$ with $\alpha \in K^*$ of infinite order or to $x \mapsto x+1$ (only when $\mathrm{char}(\mathbf{K}) = 0$). By the algebraic stability of f, f^{-i} is regular at p for all $i > 0$ and all the points $f^{-i}(p), i > 0$ are on distinct fibres. Denote by x_0, x_1 the points $\pi(p), \overline{f}(\pi(p))$. By Theorem 3.6, we know that the fibres F_{x_0}, F_{x_1} are not singular. Thus f is regular on $F_{x_0} \setminus \{p\}$ and contracts it onto a point $q \in F_{x_1}$; f^{-1} is regular on $F_{x_1} \setminus \{q\}$ and contracts it onto p. Now pick a point x_n in the forward orbit of x_0 by $\overline{f}$ and consider the fibre F_{x_n}. The fibre F_{x_n} cannot be contracted onto q by $f^{-(n-1)}$ because of the algebraic stability of f. As a consequence it is contracted by f^{-n} onto p. $\square$

Corollary 3.19 *Let f be a Jonquières twist acting algebraically stably on a conic bundle X as in the statement of Theorem 3.6. Suppose that the base action $\overline{f} \in \mathrm{PGL}_2(\mathbf{K})$ is of infinite order and there is an indeterminacy point of f located on a fibre $F_x \subset X$ such that $\overline{f}(x) \neq x$.*

1. *If $\overline{f}$ is of the form $x \mapsto x+1$ then $\mathrm{Cent}_b(f)$ is isomorphic to $\mathbf{Z}$;*
2. *if $\overline{f}$ is of the form $x \mapsto \alpha x$ then $\mathrm{Cent}_b(f)$ is isomorphic to the product of $\mathbf{Z}$ with a finite cyclic group.*

Note that the first case does not occur when $\mathrm{char}(\mathbf{K}) \neq 0$.

Proof Proposition 3.18 shows that the birational transformation f satisfies the hypothesis of Proposition 3.16. Let n denote the number of persistent indeterminacy points of f with infinite backward orbits. Let $g \in \mathrm{Cent}(f)$. Proposition 3.16 says that $g^{n!} \circ f^l$

preserves every member of a pencil of rational curves for some $l \in \mathbf{Z}$. The proof of Proposition 3.16 shows that certain members of this pencil of rational curves are fibres of the initial rational fibration on X, so this pencil of rational curves is the initial rational fibration. This means $\overline{g}^{n!} \circ \overline{f}^{l} = \mathrm{Id} \in \mathrm{PGL}_2(\mathbf{K})$.

When $\mathrm{char}(\mathbf{K}) = 0$ and $\overline{f}$ is $x \mapsto x+1$, its centralizer in $\mathrm{PGL}_2(\mathbf{K})$ is isomorphic to the additive group $\mathbf{K}$ and this group is torsion free. Thus, $\mathrm{Cent}_b(f)$ is contained in an infinite cyclic group in which $< \overline{f} >$ is of index $\leq n!$. The conclusion follows in this case.

When $\overline{f}$ is $x \mapsto \alpha x$ with α of infinite order, its centralizer in $\mathrm{PGL}_2(\mathbf{K})$ is isomorphic to the multiplicative group $\mathbf{K}^*$. The difference is that, in this case it is possible that $\overline{g}$ is of finite order $\leq n!$. Thus, we may have an additional finite cyclic factor of $\mathrm{Cent}_b(f)$.$\square$

3.4 Local analysis around a fibre

Now we need to study the case where there is no persistent indeterminacy points. In this section we will work in the following setting:

- Let f be a base-wandering Jonquières twist. We can suppose that $\overline{f}$ is $x \mapsto \alpha x$ or $x \mapsto x+1$.
- Up to taking an algebraically stable model as in Theorem 3.6, we can suppose that f is a birational transformation of a conic bundle X which satisfies the properties in Theorem 3.6.
- We assume that the only indeterminacy points of f are on the fibres F_0, F_∞.

Without loss of generality, let us suppose that f has an indeterminacy point p on the fibre F_∞. By algebraic stability f^{-1} has an indeterminacy point $q \neq p$ on F_∞. If $x \in \mathbb{P}^1$ is not 0 nor ∞, then the orbit of x under $\overline{f}$ is infinite and the fibre F_x is regular. As f has an indeterminacy point on F_∞, the fibre F_∞ is also regular. Assume that F_0 is singular, then it is the union of two (-1)-curves and f exchanges the two components. Since the aim of this section is to prove that $\mathrm{Cent}_b(f)$ is finite by cyclic, it is not harmful to replace f with f^2 so that the two components of F_0 are no more exchanged and we can assume that F_0 is regular. Thus, we can suppose that

- the surface X is a Hirzebruch surface.

If $\overline{f}$ is $x \mapsto \alpha x$, then $\mathrm{Cent}_b(f)$ is contained in $\{(x \mapsto \gamma x), \gamma \in \mathbf{K}^*\}$ and all elements of $\mathrm{Cent}_b(f)$ fix 0 and ∞. Similarly if $\overline{f}$ is $x \mapsto x+1$ then all elements of $\mathrm{Cent}_b(f)$ fix ∞. Thus F_0 or F_∞ is $\mathrm{Cent}(f)$-invariant (under total transforms), we will study the (semi-)local behaviour of the elements in $\mathrm{Cent}(f)$ around such an invariant fibre.

3.4.1 An infinite chain

We blow up X at p, q the indeterminacy points of f, f^{-1}, obtaining a new surface X_1. The fibre of X_1 over 0 is a chain of three rational curves $C_{-1} + C_0 + C_1$ where C_1 (resp. C_{-1}) is the exceptional curve corresponding to p (resp. q) and C_0 is the strict transform of $F_\infty \subset X$. Now f induces a birational transformation f_1 of X_1. As in Lemma 3.7, we know that f_1 (resp. f_1^{-1}) has an indeterminacy point p_2 (resp. q_2) on C_1 (resp. C_{-1}) which is disjoint from C_0. We then blow up p_2, q_2 and repeat the process. We have:

- for every $n \in \mathbf{N}$, a surface X_n on which f induces a birational transformation f_n;
- the fibre of X_n over 0 is a chain of rational curves $C_{-n}, \cdots, C_0, \cdots, C_n$;
- f_n (resp. f^{-n}) has an indeterminacy point p_{n+1} (resp. q_{n+1}) on C_n (resp. C_{-n}) disjoint from C_{n-1} (resp. $C_{-(n-1)}$).

Let g be a birational transformation of X which commutes with f. We already observed that F_∞ is an invariant fibre of g. If g is regular on F_∞, then the commutativity implies that g preserves the set $\{p,q\}$. Suppose that g is not regular on F_∞. Then g (resp. g^{-1}) has an indeterminacy point p' (resp. q') on F_∞. Replacing g by g^{-1} or f by f^{-1}, we can suppose that $p' \neq q$. Then for every point $x \in F_\infty$ such that $x \neq p, p'$, we have that $g(q) = g(f(x)) = f(g(x))$ is a point, thus equals q. This further implies $q = q'$. Then we apply the same argument to g, f^{-1}, obtaining $p = p'$. In summary, g is either regular on F_∞ and preserves $\{p,q\}$, or the set of indeterminacy points of g, g^{-1} on F_∞ is exactly $\{p,q\}$.

We lift g to a birational transformation on X_n. By repeating the above arguments, we deduce that for all $n \in \mathbf{N}$ the two indeterminacy points of f_n, f_n^{-1} on the fibre $F_\infty \subset X_n$ coincide with that of g_n, g_n^{-1} if the later exist. This means that for a C_i given, and for sufficiently large n, the rational curve C_i is a component of the fibre of X_n and g_n maps it to another component C_j of the fibre. In other words g acts on the infinite chain of rational curves $\sum_{n \in \mathbf{Z}} C_n$. The dual graph of this infinite chain of rational curves is a chain of vertices indexed by $\mathbf{Z}$. The action of f on the dual graph is just a non trivial translation. The isomorphism group of the dual graph is isomorphic to $\mathbf{Z} \rtimes \mathbf{Z}/2\mathbf{Z}$. Those isomorphisms which commute with a non trivial translation coincide with the subgroup of translations $\mathbf{Z}$. The above considerations can be summarized as follows:

Lemma 3.20 *There is a group homomorphism $\Phi : \mathrm{Cent}(f) \to \mathbf{Z}$ such that $g(C_n) = C_{\Phi(g)+n}$ for $g \in \mathrm{Cent}(f)$. An element $g \in \mathrm{Cent}(f)$ is in the kernel of Φ if and only if $g(C_n) = C_n$ for every $n \in \mathbf{Z}$. In other words an element g of the kernel of Φ is regular on the fibre F_∞ and fixes the indeterminacy points of f, f^{-1} on this fibre.*

Lemma 3.21 *Let g be an element of $\mathrm{Cent}(f)$. Let $x \in \mathbb{P}^1$ be a point not fixed by $\overline{f}$. Then g can not have any indeterminacy points on the fibre F_x over x.*

Proof By our hypothesis f is regular on all fibres F_{x_n} where $\{x_n, n \in \mathbf{Z}\}$ denote the orbit of x under $\overline{f}$. If g had an indeterminacy point p on F_x, then $f(p), f^2(p), \cdots$ would give us an infinite number of indeterminacy points of g. $\square$

Corollary 3.22 *Suppose that $\overline{f}$ is conjugate to $x \mapsto x+1$ (in particular $\mathrm{char}(\mathbf{K}) = 0$). Let $g \in \mathrm{Cent}(f)$ be in the kernel of $\Phi : \mathrm{Cent}(f) \to \mathbf{Z}$. Then g is an automorphism of X. Furthermore g preserves the rational fibration fibre by fibre.*

Proof Lemma 3.20 says that g does not have any indeterminacy point on the fibre F_∞. Lemma 3.21 says that g does not have any indeterminacy point elsewhere neither.

Thus, g is an automorphism. Since $\overline{g}$ commutes with $\overline{f}: x \mapsto x+1$, $\overline{g}$ is $x \mapsto x+v$ for some $v \in \mathbf{K}$. Suppose by contradiction that $v \neq 0$. Then g is an elliptic element of infinite order and $f \in \mathrm{Cent}(g)$. We can apply Theorem 2.7 to g, f and put them in normal form. As f is a Jonquières twist, the rational fibration preserved simultaneously by f and g is unique and it must be the rational fibration appeared in the normal form. Hence, Theorem 2.7 forbids $\overline{f}, \overline{g}$ to be both non-trivial and of the form $x \mapsto x + sth$. $\square$

When $\overline{f}$ is of the form $x \mapsto \alpha x$, there are two special fibres F_0, F_∞ and the above easy argument does not work.

3.4.2 Formal considerations along a fibre

In the rest of this section we will assume that $\overline{f}$ is $x \mapsto \alpha x$. There are two invariant fibers F_∞ and F_0 in this case. We assume that f has an indeterminacy point q on F_0.

The idea of what we do in the sequel is as follows. Let us look at the case where $\mathbf{K} = \mathbf{C}$. The indeterminacy point $q \in F_0$ of f^{-1} is a fixed point of f, at which the differential of f has two eigenvalues 0 and α; the fibre directon is superattracting and in the transverse direction f is just $x \mapsto \alpha x$. Therefore there is a local invariant manifold at q for f, which is a local holomorphic section of the rational fibration. Likewise, there is a local invariant manifold at $p \in F_0$, the indeterminacy point of f. These two local holomorphic sections allow us to conjugate locally holomorphically f to $(\alpha x, a(x)y)$ where a is a germ of holomorphic function. The structure of Jonquières maps is nice enough to allow us to apply this geometric idea over any field in an elementary way. We need just to work with formal series instead of polynomials.

From now on we fix $f: (x,y) \dashrightarrow (\alpha x, \frac{A(x)y+B(x)}{C(x)y+D(x)})$ where $\alpha \in \mathbf{K}^*$ is of infinite order and $A, B, C, D \in \mathbf{K}[x]$. Without loss of generality, we suppose that 1) the point $(0,0)$ (resp. $(0,\infty)$) is an indeterminacy point of f (resp. f^{-1}); 2) one of the A, B, C, D is not a multiple of x. This implies

$$B(0) = C(0) = D(0) = 0,\ A(0) \neq 0. \tag{3}$$

We will consider A, B, C, D as elements of the ring of formal series $\mathbf{K}[\![x]\!]$. We will also view f as an element of the formal Jonquières group $\mathrm{PGL}_2(\mathbf{K}((x))) \rtimes \mathbf{K}^*$ whose elements are formal expressions of the form $(\mu x, \frac{a(x)y+b(x)}{c(x)y+d(x)})$ where $\mu \in \mathbf{K}^*$ and a, b, c, d belong to $\mathbf{K}((x))$, the fraction field of $\mathbf{K}[\![x]\!]$.

Normal form. We want to conjugate f to a formal expression of the form $(\alpha x, \beta(x)y), \beta \in \mathbf{K}((x))$ by some formal expression $(x, \frac{E(x)y+F(x)}{G(x)y+H(x)})$ with $E, F, G, H \in \mathbf{K}[\![x]\!]$. This amounts to say that we are looking for $E, F, G, H \in \mathbf{K}[\![x]\!]$ such that $EF - GH \neq 0$ and

$$\begin{pmatrix} E(\alpha x) & F(\alpha x) \\ G(\alpha x) & H(\alpha x) \end{pmatrix}^{-1} \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix} \begin{pmatrix} E(x) & F(x) \\ G(x) & H(x) \end{pmatrix}$$

is a diagonal matrix. By writing out the explicit expressions of the up-right entry and the down-left entry of this matrix product, we obtain two equations to solve:

$$F(x)H(\alpha x)A(x) + H(x)H(\alpha x)B(x) - F(x)F(\alpha x)C(x) - H(x)F(\alpha x)D(x) = 0 \tag{4}$$

$$-E(x)G(\alpha x)A(x) - G(x)G(\alpha x)B(x) + E(x)E(\alpha x)C(x) + G(x)E(\alpha x)D(x) = 0 \tag{5}$$

We will use minuscules to denote the coefficients of the formal series, e.g. $E(x) = \sum_{i\in\mathbf{N}} e_i x^i$. Let us first look at the constant terms of equations (4), (5), they give

$$-e_0g_0a_0 - g_0^2b_0 + e_0^2c_0 + e_0g_0d_0 = 0 = f_0h_0a_0 + h_0^2b_0 - f_0f_0c_0 - f_0h_0d_0.$$

Since $b_0 = c_0 = d_0 = 0$ and $a_0 \neq 0$ (see Equation (3)), we must have $e_0g_0 = f_0h_0 = 0$. We can choose $f_0 = g_0 = 0$ and $e_0 = h_0 = 1$, this guarantees in particular that our solution will satisfy $EH - FG \neq 0$.

Remark that the equations (4) and (5) involve respectively only E,G and F,H, and they have exactly the same form. So it suffices to show the existence of E,G which satisfy equation (4). The constant term is done, let us look at the x term. This leads to a linear equation in e_1, g_1 with coefficients involving $a_0, b_0, c_0, d_0, e_0, g_0$ and α. Therefore there exists at least one solution for e_1, g_1. Then we turn to the next term and get a linear equation in e_2, g_2, and so on. Hence, we can find E,F,G,H which satisfy the desired properties. To sum up, we have:

Lemma 3.23 *There exists $E,F,G,H \in \mathbf{K}[\![x]\!]$ such that:*

- $E(0) = H(0) = 1$ *and* $F(0) = G(0) = 0$*, in particular* $\begin{pmatrix} E & F \\ G & H \end{pmatrix} \in \mathrm{PGL}_2(\mathbf{K}((x)))$*;*
- $(x, \frac{E(x)y+F(x)}{G(x)y+H(x)})$ *conjugates* f *to* $(\alpha x, \beta(x)y)$ *for some* $\beta \in \mathbf{K}((x))$*;*

Projective line over $\mathbf{K}((x))$. We call an element of $\mathbb{P}^1(\mathbf{K}((x))) = \mathbf{K}((x)) \bigcup \{\infty\}$ a formal section. We say a formal section $\theta(x)$ passes through the origin if $\theta(0) = 0$. An element $u = (\mu x, \frac{a(x)y+b(x)}{c(x)y+d(x)})$ of the formal Jonquières group $\mathrm{PGL}_2(\mathbf{K}((x))) \rtimes \mathbf{K}^*$ acts on $\mathbb{P}^1(\mathbf{K}((x)))$ in the following way:

$$\theta(x) \mapsto u\cdot\theta(x) = \begin{cases} \infty & \text{if } c(\mu^{-1}x)\theta(\mu^{-1}x) + d(\mu^{-1}x) = 0 \\ \frac{a(\mu^{-1}x)\theta(\mu^{-1}x)+b(\mu^{-1}x)}{c(\mu^{-1}x)\theta(\mu^{-1}x)+d(\mu^{-1}x)} & \text{otherwise} \end{cases},$$

$$\infty \mapsto \begin{cases} \infty & \text{if } c = 0 \\ \frac{a(\mu^{-1}x)}{c(\mu^{-1}x)} & \text{if } c \neq 0 \end{cases}.$$

Geometrically this is saying that a formal section of the rational fibration is sent to another by a formal Jonquières transformation. Remark that this action on $\mathbb{P}^1_{\mathbf{K}((x))}$ is not an automorphism of $\mathbf{K}((x))$-algebraic variety. In scheme theoretic language, we have a commutative diagram:

$$\begin{array}{ccc} \mathbb{P}^1_{\mathbf{K}((x))} & \xrightarrow{\theta\mapsto u\cdot\theta} & \mathbb{P}^1_{\mathbf{K}((x))} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbf{K}((x))) & \xrightarrow{\mu x \leftarrow x} & \mathrm{Spec}(\mathbf{K}((x))). \end{array}$$

Thus, we have a group homomorphism from $\mathrm{PGL}_2(\mathbf{K}((x))) \rtimes \mathbf{K}^*$ to the group of such twisted automorphisms of $\mathbb{P}^1_{\mathbf{K}((x))}$.

Now let $g \in \mathrm{Cent}(f)$ be an element in the kernel of Φ. Recall (see Lemma 3.20) that g is regular on the fibre F_0 and fixes $(0,0),(0,\infty)$. We showed that f is conjugate by $\begin{pmatrix} E & F \\ G & H \end{pmatrix}$ to a formal expression $\hat{f}$ of the form $(\alpha x, \beta(x)y)$. We conjugate g by $\begin{pmatrix} E & F \\ G & H \end{pmatrix}$ too to get a formal expression $\hat{g}$. Then $\hat{g}$ commutes with $\hat{f}$.

Recall that, by Lemma 3.23, we get $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ when we evaluate the formal expression $\begin{pmatrix} E & F \\ G & H \end{pmatrix}$ at $x=0$. Together with the fact that $g \in \mathrm{Ker}(\Phi)$, this implies that we get $y \mapsto \delta_0 y$ for some $\delta_0 \in \mathbf{K}^*$ when we evaluate $\hat{g}$ at $x=0$.

Let us consider the actions of $\hat{f}, \hat{g}$ on $\mathbb{P}^1_{\mathbf{K}((x))}$ as described above. Since $\hat{f}$ is in diagonal form, it fixes the points 0 and ∞ of $\mathbb{P}^1_{\mathbf{K}((x))}$.

Lemma 3.24 *If $\theta \in \mathbb{P}^1_{\mathbf{K}((x))}$ satisfies $\theta(0)=0$ and $\hat{f} \cdot \theta(x) = \theta(x)$, then $\theta = 0$.*

Proof The equation $\hat{f} \cdot \theta(x) = \theta(x)$ writes as $\beta(\alpha^{-1}x)\theta(\alpha^{-1}x) = \theta(x)$, i.e. $\theta(\alpha x)^{-1}\beta(x)\theta(x) = 1$. Suppose by contradiction that θ is not 0. Then we can write $\theta(x)$ as $x^r\tilde{\theta}(x)$ where $r>0$ and $\tilde{\theta}(0) \neq 0$. Hence we have $\tilde{\theta}(\alpha x)^{-1}\beta(x)\tilde{\theta}(x) = \alpha^r$. This implies that $\hat{f}$ is conjugate by $(x, \tilde{\theta}(x)y)$ to $(\alpha x, \alpha^r y)$. Since $\tilde{\theta}(0) \neq 0$ and $\begin{pmatrix} E(0) & F(0) \\ G(0) & H(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, this implies that the initial Jonquières twist f is regular on the fibre F_0, contradiction.$\square$

Since $\hat{g}$ is $y \mapsto \delta_0 y$ at $x=0$, it sends the formal section $0 \in \mathbb{P}^1(\mathbf{K}((x)))$ to another former section passing through the origin. The fact that $\hat{f}$ and $\hat{g}$ commute and the fact that 0 is the only fixed formal section of $\hat{f}$ which passes through the origin imply that $\hat{g}$ fixes $0 \in \mathbb{P}^1_{\mathbf{K}((x))}$. Likewise $\hat{g}$ fixes ∞ too. Therefore $\hat{g}$ writes as $(\gamma x, \delta(x)y)$ where $\gamma \in \mathbf{K}^*$ and $\delta \in \mathbf{K}((x))$ satisfies $\delta(0) = \delta_0 \neq 0$.

Normal forms for a pair. Let us assume for the moment that γ is not a root of unity; we are going to prove that this is impossible. We want to, under this hypothesis, conjugate $\hat{g} = (\gamma x, \delta(x)y)$ to $(\gamma x, \delta(0)y)$ by $h = (x, \xi(x)y)$ for some $\xi \in \mathbf{K}[\![x]\!]$. Remark that the conjugate of $\hat{f}$ by h will still be in diagonal form.

We write $\delta = \frac{\omega}{\sigma}$ where $\omega, \sigma \in \mathbf{K}[\![x]\!]$ satisfies $\omega(0) \neq 0, \sigma(0) \neq 0$ and $\frac{\omega(0)}{\sigma(0)} = \delta(0)$. We will write ξ as $\sum_{i \in \mathbf{N}} \xi_i x^i$, and likewise for σ, ω.

After conjugation by $h = (x, \xi(x)y)$, $\hat{g}$ becomes

$$\tilde{g} = h \circ \hat{g} \circ h^{-1} = (\gamma x, \frac{\xi(\gamma x)}{\xi(x)} \frac{\omega(x)}{\sigma(x)} y).$$

Therefore the equation we want to solve is

$$\xi(\gamma x)\omega(x) = \frac{\omega_0}{\sigma_0}\xi(x)\sigma(x). \tag{6}$$

The constant terms of the two sides are automatically equal, let us just choose $\xi_0 = 1$. Comparing the other terms, we obtain

$$\xi_0\omega_1 + \gamma\xi_1\omega_0 = \frac{\omega_0}{\sigma_0}(\xi_0\sigma_1 + \xi_1\sigma_0)$$
$$\xi_0\omega_2 + \gamma\xi_1\omega_1 + \gamma^2\xi_2\omega_0 = \frac{\omega_0}{\sigma_0}(\xi_0\sigma_2 + \xi_1\sigma_1 + \xi_2\sigma_0)$$
$$\cdots$$

which are equivalent to

$$(\gamma-1)\omega_0\xi_1 = \frac{\omega_0}{\sigma_0}\xi_0\sigma_1 - \xi_0\omega_1$$
$$(\gamma^2-1)\omega_0\xi_2 = \frac{\omega_0}{\sigma_0}(\xi_0\sigma_2 + \xi_1\sigma_1) - \xi_0\omega_2 - \gamma\xi_1\omega_1$$
$$\cdots.$$

For the i-th term, we have a linear equation whose coefficient before ξ_i is $(\gamma^i - 1)\omega_0$. Since $\omega \neq 0$ and we have supposed that γ is not a root of unity, The above equations always have solutions. In summary, we have the following intermediate lemma (we will get from this lemma a contradiction so its hypothesis is in fact absurd):

Lemma 3.25 *Suppose that $g \in \mathrm{Ker}(\Phi)$ and the action of g on the base is of infinite order. Then we can conjugate f and g, simultaneously by an element in $\mathrm{PGL}_2(\mathbf{K}((x)))$ whose evaluation at $x = 0$ is $\mathrm{Id} : y \mapsto y$, to*

$$\tilde{g} = (\gamma x, \delta y),\ \tilde{f} = (\alpha x, \beta(x)y)$$

where $\alpha, \gamma, \delta \in \mathbf{K}^*$, $\beta \in \mathbf{K}((x))^*$ and α, γ are of infinite order.

Writing down the equation $\tilde{f} \circ \tilde{g} = \tilde{g} \circ \tilde{f}$, we get $\delta\beta(x) = \delta\beta(\gamma x)$. As $\delta \neq 0$, we get $\beta(x) = \beta(\gamma x)$. We write $\beta = \frac{\beta^{num}}{\beta^{den}}$ with $\beta^{num}, \beta^{den} \in \mathbf{K}[[x]]$ such that at least one of the $\beta_0^{num}, \beta_0^{den}$ is not 0. The equation becomes

$$\beta^{num}(x)\beta^{den}(\gamma x) = \beta^{den}(x)\beta^{num}(\gamma x).$$

By comparing the coeffcients of two sides, we get

$$\forall k \in \mathbf{N},\ \sum_{i+j=k} \beta_i^{num}\beta_j^{den}\gamma^j = \sum_{i+j=k} \beta_i^{den}\beta_j^{num}\gamma^j.$$

Then by induction on k we get from these equations:

1. either $\beta^{num} = 0$ (when $\beta_0^{num} = 0$), this is impossible;
2. or $\beta^{den} = 0$ (when $\beta_0^{den} = 0$), this is again impossible;
3. or $\beta^{num} = \kappa\beta^{den}$ for some $\kappa \in \mathbf{K}^*$ (when $\beta_0^{num}\beta_0^{den} \neq 0$). Then $\tilde{f} = (\alpha x, \kappa y)$, this contradicts the fact that the original birational transformation f has an indeterminacy point on the fibre F_0 because to get $\tilde{f}$ we only did conjugations whose evaluation at $x = 0$ are the identity $y \mapsto y$.

Thus, we get

Proposition 3.26 *Suppose that $g \in \mathrm{Ker}(\Phi)$. Then $\overline{g}$ is of finite order and g is an elliptic element of $\mathrm{Cr}_2(\mathbf{K})$.*

Proof We have already showed that $\overline{g}$ can not be of infinite order. Then an iterate g^k is in $\mathrm{Jonq}_0(\mathbf{K})$ and $f \in \mathrm{Cent}(g^k)$. By Theorem 2.14, an element which commutes with a Jonquières twist in $\mathrm{Jonq}_0(\mathbf{K})$ can not have an infinite action on the base. As $\overline{f}$ is of infinite order, g^k must be elliptic. So g must be elliptic. □

3.4.3 Another fibre

The base action $\overline{f} \in \mathrm{PGL}_2(\mathbf{K})$ is $x \mapsto \alpha x$, it has two fixed points 0 and ∞. Recall that we are always under the hypothesis that the indeterminacy points of f are on the fibres F_0, F_∞. We have done analysis around the fibre F_0 on which f has an indeterminacy point. We will denote by Φ_0 the homomorphism Φ we considered before. In case f has also an indeterminacy point on F_∞, we denote the corresponding homomorphism by Φ_∞. We are going to reduce the proof to a situation where the following lemma applies.

Lemma 3.27 *The image of $\mathrm{Aut}(X) \bigcap \mathrm{Ker}(\Phi_0) \subset \mathrm{Cent}(f)$ in $\mathrm{Cent}_b(f) \subset \mathrm{PGL}_2(\mathbf{K})$ is a finite cyclic group.*

Proof We recall first that the automorphism group of a Hirzebruch surface is an algebraic group (see [Mar71]). An element of $\mathrm{Cent}(f)$ which is regular everywhere on H must be in $\mathrm{Ker}(\Phi_0)$. Thus, $\mathrm{Aut}(X) \bigcap \mathrm{Ker}(\Phi_0) = \mathrm{Aut}(X) \bigcap \mathrm{Cent}(f)$ is an algebraic subgroup of $\mathrm{Aut}(H)$. An automorphism of a Hirzebruch surface always preserves the rational fibration and there is a morphism of algebraic groups from $\mathrm{Aut}(X)$ to $\mathrm{PGL}_2(\mathbf{K})$ (see [Mar71]). The image of $\mathrm{Aut}(X) \bigcap \mathrm{Ker}(\Phi_0) \subset \mathrm{Cent}(f)$ in $\mathrm{Cent}_b(f) \subset \mathrm{PGL}_2(\mathbf{K})$ is an algebraic subgroup Λ of $\mathrm{PGL}_2(\mathbf{K})$. By Proposition 3.26, the elements of Λ are all multiplication by roots of unity. If Λ was infinite then it would equal to its Zariski closure in $\mathrm{PGL}_2(\mathbf{K})$ and would be isomorphic to the multiplicative group $\mathbf{K}^*$. But the existence of a base-wandering Jonquières twist means that $\mathbf{K}^*$ contains elements of infinite order, for example α. This contradicts the fact that $\Lambda = \mathbf{K}^*$ is of torsion. The conclusion follows. □

We first look at the case where we have two homomorphisms Φ_0, Φ_∞ to use:

Proposition 3.28 *If f has an indeterminacy point on F_∞, then $\mathrm{Ker}(\Phi_0) = \mathrm{Ker}(\Phi_\infty)$ is a subgroup of $\mathrm{Aut}(X)$. The image of $\mathrm{Ker}(\Phi_0)$ in $\mathrm{Cent}_b(f) \subset \mathrm{PGL}_2(\mathbf{K})$ is a finite cyclic group.*

Proof Let g be an element of $\mathrm{Ker}(\Phi_0)$. By Proposition 3.26 g is an elliptic element of $\mathrm{Cr}_2(\mathbf{K})$. If $\Phi_\infty(g)$ were not trivial, then g would act by a non trivial translation on the corresponding infinite chain of rational curves and could not be conjugate to any automorphism. This means g must belong to $\mathrm{Ker}(\Phi_\infty)$ and consequently g must be an automorphism of H. The second part of the statement follows from Lemma 3.27. □

When f is regular on F_∞, we may need to do a little more, but we get more precise information as well:

Proposition 3.29 *If f has no indeterminacy points on F_∞, then* $\mathrm{Ker}(\Phi_0)$ *is a finite cyclic group whose elements are automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$ of the form $(x,y) \mapsto (\gamma x, y)$ with γ a root of unity.*

Proof Assume that f is regular on F_∞. Let $g \in \mathrm{Ker}(\Phi_0)$ be a non trivial element, it is regular on F_0. By Lemma 3.21, an indeterminacy point of g can only be located on F_∞.

Suppose that g has an indeterminacy point p on F_∞. Then g^{-1} also has an indeterminacy point q on F_∞. If $p \neq q$, then g would act by translation on the corresponding infinite chain of rational curves. This means that g would never be conjugate to an automorphism of some surface and contradicts Proposition 3.26 which asserts that g is elliptic. Thus, we have $p = q$. The facts that f commutes with g and that f is regular on F_∞ imply $f(p) = p$. We blow up the Hirzebruch surface X at p to get a new surface X' and induced actions f', g'. The induced action f' is still regular on the fibre F'_∞ and preserves both of the two irreducible components. If g' has an indeterminacy point on F'_∞, then as before it coincides with the indeterminacy point of g'^{-1} and must be fixed by f'. Then we can keep blowing up indeterminacy points of maps induced from g, or contracting g-invariant (-1)-curves in the fibre, without loosing the regularity of the map induced by f. As g is elliptic, we will get at last a surface $\hat{X}$ with induced actions $\hat{f}, \hat{g}$ which are all regular on the fibre over ∞. We can suppose that $\hat{X}$ is minimal among the surfaces with this property. In particular $\hat{g}$ is an automorphism of $\hat{X}$. Moreover, the proof of Theorem 3.6 shows that $\hat{X}$ is a conic bundle and the only possible singular fibre is $\hat{F}_\infty$. We claim that $\hat{F}_\infty$ is in fact regular. Suppose by contradiction that $\hat{F}_\infty$ is singular. Then it is a chain of two (-1)-curves and $\hat{g}$ exchanges the two components. However the conic bundle $\hat{X}$ is obtained from a Hirzebruch surface by a single blow-up, it has a unique section of negative self-intersection which passes through only one of the two components of the singular fibre. As a consequence, the automorphism $\hat{g}$ can not exchange the two components, contradiction. Thus, replacing X by $\hat{X}$, we can suppose from the beginning that g is an automorphism of the Hirzebruch surface X.

Suppose by contradiction that g preserves only finitely many sections of the rational fibration. Since f commutes with g, we can assume, after perhaps replacing f by some of its iterates, that f and g preserve simultaneously a section of the rational fibration. Removing this section and the fibre F_0 from H, we get an open set isomorphic to $\mathbb{A}^2$ restricted to which f writes as $(x', y') \mapsto (\alpha^{-1}x', A(x')y' + B(x'))$ where $A, B \in \mathbf{K}(x')$. The rational function A must be a constant because f acts as an automorphism on this affine open set. Likewise the rational function B must be a polynomial. But then $(deg(f^n))_{n\in\mathbf{N}}$ would be a bounded sequence. This contradicts the fact that f is a Jonquières twist.

Hence, if $g \in \mathrm{Ker}(\Phi)$ is non-trivial then it preserves necessarily infinitely many sections. This forces g to preserve each member of a pencil of rational curves on X whose general members are sections (see Lemma 3.17). This is only possible if $X = \mathbb{P}^1 \times \mathbb{P}^1$ and g acts as $(x,y) \mapsto (\gamma x, y)$ with $\gamma \in \mathbf{K}^*$; here the projection of $\mathbb{P}^1 \times \mathbb{P}^1$ onto the first factor is the original rational fibration we were looking at. This allows us to conclude by Lemma 3.27. □

Example 3.30 Let μ be a k-th root of unity, the pair $f:(x,y)\mapsto(\alpha x,\frac{(1+x^k)y+x^k}{(2+x^k)y+1+x^k}), g:(x,y)\mapsto(\mu x,y)$ satisfy the conditions in Proposition 3.29.

Now let f be a base-wandering Jonquières twist which satisfies the hypothesis made at the beginning of Section 3.4; in particular f is regular outside $F_0\bigcap F_\infty$ and $\overline{f}$ is $x\mapsto\alpha x$ or $x\mapsto x+1$. The image $\Phi_\infty(\mathrm{Cent}(f))$ is an infinite cyclic subgroup of $\mathbf{Z}$ and is isomorphic to $\mathbf{Z}$, it is generated by $\Phi_\infty(g)$ for some $g\in\mathrm{Cent}(f)$. Then for any $h\in\mathrm{Cent}(f)$, there exists $k\in\mathbf{Z}$ such that $g^{-k}\circ h\in\mathrm{Ker}(\Phi_\infty)$. Thus, $\overline{g}^{-k}\circ\overline{h}$ belongs to the image of $\mathrm{Ker}(\Phi_\infty)$ in $\mathrm{Cent}_b(f)$. By Corollary 3.22, Proposition 3.28 and Proposition 3.29, the image of $\mathrm{Ker}(\Phi_\infty)$ in $\mathrm{Cent}_b(f)$ is at worst finite cyclic. Note that $\mathrm{Cent}_b(f)$ is always abelian. Therefore we obtain the last piece of information to prove Theorem 3.2:

Proposition 3.31 *Let f be a base-wandering Jonquières twist which satisfies the hypothesis made at the beginning of Section 3.4. Let g be an element of* $\mathrm{Cent}(f)$ *such that* $\Phi_0(g)$ *generates the image of* Φ*. Then* $\mathrm{Cent}_b(f)$ *is the product of a finite cyclic group with the infinite cyclic group generated by* $\overline{g}$*.*

4 Proofs of the main results

Proof (of Theorem 1.1) Centralizers of loxodromic elements are virtually cyclic by Theorem 2.1 of Blanc-Cantat. It is proved in [Giz80],[Can11] that centralizers of Halphen twists are virtually abelian (see Theorem 2.2). Centralizers of Jonquières twists whose actions on the base are of finite order are contained in tori over the function field $\mathbf{K}(x)$, thus are abelian ([CD12b] see Theorem 2.14). Our Theorem 3.3 says that centralizers of base-wandering Jonquières twists are virtually abelian. Centralizers of infinite order elliptic elements (due to [BD15]) are described in Theorem 2.7, from which we see directly that the only infinite order elliptic elements which admit non virtually abelian centralizers are those given here. □

Proof (of Theorem 1.2) The proof is a direct combination of Theorems 2.1, 2.2, 2.7, 2.14 and 3.2. □

Proof (of Remark 1.4) In the first case Γ is an elliptic subgroup, so the degree function is bounded.

In the second case, the two Halphen twists f and g are automorphisms of a rational surface X preserving an elliptic fibration $X\to\mathbb{P}^1$. The elliptic fibration is induced by the linear system corresponding to mK_X for some $m\in\mathbf{N}^*$. For $n\in\mathbf{N}$, the actions of f^n and g^n on $Pic(X)$ are respectively

$$D\mapsto D-mn(D\cdot K_X)\Delta_i+\left(-\frac{m^2}{2}(D\cdot K_X)\cdot(n\Delta_i)^2+m(D\cdot(n\Delta_i))\right)K_X,\quad i=1,2$$

where $(\cdot)$ denotes the intersection form and $\Delta_i\in Pic(X)$ satisfies $\Delta_i\cdot K_X=0$ (cf. [Giz80],

[BD15]). Therefore the action of $f^i \circ g^j$ on $Pic(X)$ is

$$D \mapsto D - mi(D \cdot K_X)\Delta_1 - mj(D \cdot K_X)\Delta_2 + \lambda_{ij}K_X \quad \text{where}$$
$$\lambda_{ij} = -\frac{m^2}{2}(D \cdot K_X) \cdot \left(i^2\Delta_1^2 + j^2\Delta_2^2\right) + mD \cdot (i\Delta_1 + j\Delta_2) - ijm^2(D \cdot K_X)(\Delta_1 \cdot \Delta_2).$$

Let Λ be an ample class on X. Then the degree of $f^i \circ g^j$ is up to a bounded term (cf. [BD15] Section 5)

$$\Lambda \cdot (f^i \circ g^j)^* \Lambda = \Lambda^2 - \frac{m^2}{2}(\Lambda \cdot K_X)^2 \left(i^2\Delta_1^2 + j^2\Delta_2^2\right) - ijm^2(\Lambda \cdot K_X)^2(\Delta_1 \cdot \Delta_2).$$

Note that Δ_1^2 and Δ_2^2 are negative.

Let us consider the third case. Firstly assume that $\Gamma \cap \mathrm{Jonq}_0(\mathbf{K})$ is contained in a split torus over $\mathbf{K}(x)$. Then up to conjugation we can find two generators $f_0 : (x,y) \dashrightarrow (x, \frac{P(x)}{Q(x)}y), g_0 : (x,y) \dashrightarrow (x, \frac{R(x)}{S(x)}y)$ of $\Gamma \cap \mathrm{Jonq}_0(\mathbf{K})$ such that $P,Q,R,S \in \mathbf{K}[x]$ do not have common factors. If f_0 is elliptic, then $Q = 1$ and $P \in \mathbf{K}$, so the degree of $f_0^i g_0^j$ is $|j|(deg(R) + deg(S)) + 1$. If f_0, g_0 are both Jonquières twists, then the degree of $f_0^i g_0^j$ is $|i|(deg(P) + deg(Q)) + |j|(deg(R) + deg(S)) + 1$. Now assume that $\Gamma \cap \mathrm{Jonq}_0(\mathbf{K})$ is contained in a non-split torus over $\mathbf{K}(x)$. The torus becomes split over a quadratic extension L of $\mathbf{K}(x)$. The field L is the function field of a double cover of $\mathbb{P}^1$, it has also a notion of degree. On $\mathbf{K}(x)$, the L-degree function is a multiple of the $\mathbf{K}(x)$-degree function. Therefore the arguments in the split case still work.

In the fourth case the description of the degree function follows directly from the explicit expressions. □

Theorem 4.1 *Let $G \subset \mathrm{Cr}_2(\mathbf{K})$ be a maximal abelian subgroup which has at least one element of infinite order. Then up to conjugation one of the following possibilities holds:*

1. *G is $\{(x,y) \mapsto (\alpha x, \beta y) | \alpha, \beta \in \mathbf{K}^*\}$, $\{(x,y) \mapsto (\alpha x, y+v) | \alpha \in \mathbf{K}^*, v \in \mathbf{K}\}$ or $\{(x,y) \mapsto (x+u, y+v) | u, v \in \mathbf{K}\}$;*
2. *G is the product of $\{(x,y) \mapsto (x, \beta y) | \beta \in \mathbf{K}^*\}$ with an infinite torsion group G_1. Each element of G_1 is of the form*
$$(x,y) \dashrightarrow \left(\eta(x), y\frac{S(x)}{S(\eta(x))}\right) \text{ with } \eta \in \mathrm{PGL}_2(\mathbf{K}), S \in \mathbf{K}(x)$$
and the morphism from G_1 to $\mathrm{PGL}_2(\mathbf{K})$ embeds G_1 as a subgroup of the group of roots of unity of $\mathbf{K}$ or a subgroup of the additive group $\mathbf{K}$. All elements of G are elliptic but G is not conjugate to a group of automorphisms of any rational surface.
3. *G has a finite index subgroup contained in $\mathrm{Jonq}_0(\mathbf{K}) = \mathrm{PGL}_2(\mathbf{K}(x))$.*
4. *A finite index subgroup G' of G is a cyclic group generated by a base-wandering Jonquières twist.*

5. *A finite index subgroup G' of G is isomorphic to $\mathbf{K}^* \times \mathbf{Z}$ (resp. $\mathbf{K} \times \mathbf{Z}$) where the first factor is $\{(x,y) \mapsto (x, \beta y) | \beta \in \mathbf{K}^*\}$ (resp. $\{(x,y) \mapsto (x, y+v) | v \in \mathbf{K}\}$) and the second factor is generated by a base-wandering Jonquières twist, as in the fourth case of Theorem 1.2;*

6. *A finite index subgroup G' of G is isomorphic to $\mathbf{Z}^s$ with $s \leq 8$ and G' preserves fibrewise an elliptic fibration;*

7. *A finite index subgroup G' of G is a cyclic group generated by a loxodromic element.*

The existence of a type two maximal abelian group is less obvious than the others. We give here two examples.

Example 4.2 Let $q \in \mathbf{N}^*$. Let $(\xi_n)_n$ be a sequence of elements of $\mathbf{K}^*$ such that ξ_n is a primitive q^n-th root of unity and $\xi_n^q = \xi_{n-1}$. Let $(R_n)_n$ be a sequence of non-constant rational fractions. For $i \in \mathbf{N}$, put

$$f_{i+1} : (x,y) \dashrightarrow (\xi_{i+1}x, yS_{i+1}(x)) \text{ with } S_{i+1}(x) = \frac{R_i(x^{q^i})}{R_i(\xi_1 x^{q^i})} \frac{R_{i-1}(x^{q^{i-1}})}{R_{i-1}(\xi_2 x^{q^{i-1}})} \cdots \frac{R_1(x)}{R_1(\xi_i x)}.$$

We have $f_{i+1}^q = f_i$ for all $i \in \mathbf{N}^*$ so that the group G_1 generated by all the f_i is an infinite torsion abelian group. Let $T_i(x) = R_i(x^{q^i}) \cdots R_1(x^q)$. The conjugation by $(x,y) \dashrightarrow (x, yT_i(x))$ sends the group generated by $f_1, \cdots, f_i$ into the cyclic elliptic group $\{(x,y) \mapsto (\xi_i^j x, y) | j = 0, 1, \cdots, q^i - 1\}$. However the degree of f_i goes to infinity when i tends to infinity, which implies that G_1 can not be conjugate to an automorphism group. The product of G_1 with $\{(x,y) \mapsto (x, \beta y) | \beta \in \mathbf{K}^*\}$ is a maximal abelian subgroup of $\mathrm{Cr}_2(\mathbf{K})$; the maximality follows directly from Theorem 2.7.

Example 4.3 We can give an additive version of Example 4.2. Suppose that $\mathrm{char}(\mathbf{K}) = p > 0$. Let $(t_n)_n$ be a sequence of elements of $\mathbf{K}$ linearly independant over $\mathbf{F}_p$. Let $R \in \mathbf{K}(x)$ be a non-constant rational fraction. For $i \in \mathbf{N}$, put

$$f_{i+1} : (x,y) \dashrightarrow (x + t_{i+1}, yS_{i+1}(x)) \text{ with } S_{i+1}(x) = \frac{\prod_{(a_1,\cdots,a_i) \in \mathbf{F}_p^i} R(x - \sum_{k=1}^i a_k t_k)}{\prod_{(a_1,\cdots,a_i) \in \mathbf{F}_p^i} R(x + t_{i+1} - \sum_{k=1}^i a_k t_k)}.$$

Let G_1 be the group generated by all the f_i. The product of G_1 with $\{(x,y) \mapsto (x, \beta y) | \beta \in \mathbf{K}^*\}$ is a maximal abelian subgroup of $\mathrm{Cr}_2(\mathbf{K})$.

Proof (of Theorem 4.1) Let G be a maximal abelian subgroup of $\mathrm{Cr}_2(\mathbf{K})$. Note that if f is a non-trivial element of G, then G is the maximal abelian subgroup of $\mathrm{Cent}(f)$.

If G contains a loxodromic element f, then G is included in $\mathrm{Cent}(f)$ and is virtually the cyclic group generated by f by Theorem 2.1; this corresponds to the last case of the above statement. If G contains a Halphen twist, then by Theorem 2.2 it is virtually a free abelian group of rank ≤ 8 which preserves fibrewise an elliptic fibration; this corresponds to the sixth case.

Assume that G contains a base-wandering Jonquières twist f. Theorem 3.2 says that $\mathrm{Cent}(f)$ is virtually isomorphic to $\mathbf{K}^* \times \mathbf{Z}$, $\mathbf{K} \times \mathbf{Z}$ or $\mathbf{Z}$. Thus the same is true for G. This correponds to the fourth and the fifth case.

Assume that G contains a non-base-wandering Jonquières twist f. Theorem 2.14 says that $\mathrm{Cent}(f)$ is virtually isomorphic to an abelian subgroup of $\mathrm{PGL}_2(\mathbf{K}(x))$, so the same is true for G. This is the third case.

In the rest of the proof we assume that G *contains only elliptic elements*. Note that G is not necessarily an elliptic subgroup because it may not be finitely generated.

Assume that $\mathrm{char}(\mathbf{K}) = 0$ and G contains an element $f : (x,y) \mapsto (\alpha x, y+1)$ with $\alpha \in \mathbf{K}^*$. By Theorem 2.7 we have

$$\mathrm{Cent}(f) = \{(x,y) \dashrightarrow (\eta(x), y+R(x)) | \eta \in \mathrm{PGL}_2(\mathbf{K}), \eta(\alpha x) = \alpha\eta(x), R \in \mathbf{K}(x), R(\alpha x) = R(x).\}$$

If α has infinite order, then $G = \mathrm{Cent}(f) = \{(x,y) \dashrightarrow (\gamma x, y+v) | \gamma \in \mathbf{K}^*, v \in \mathbf{K}\}$ and we are in the first case. Assume at first that G has an element g with an infinite action on the base of the rational fibration $(x,y) \mapsto x$. If the action of g on the base is conjugate to $x \mapsto \beta x$ with $\beta \in \mathbf{K}^*$, then up to conjugation in $\mathrm{Jonq}(\mathbf{K})$ we can suppose that g is just our initial element $f : (x,y) \dashrightarrow (\alpha x, y+1)$ (see Proposition 2.4), so that G is isomorphic to $\mathbf{K}^* \times \mathbf{K}$. If the the action of g on the base is conjugate to $x \mapsto x+1$, then by choosing an appropriate coordinate x, the two elements f and g are respectively $(x,y) \mapsto (x+1, y+R(x))$ and $(x,y) \mapsto (x, y+1)$ where R is a polynomial by Lemma 2.11. We can conjugate g and f, simultaneously by $(x,y) \dashrightarrow (x, y+S(x))$ for some $S \in \mathbf{K}[x]$, to $(x,y) \mapsto (x+1, y)$ and $(x,y) \mapsto (x, y+1)$. Then we have

$$G = \mathrm{Cent}(f) \bigcap \mathrm{Cent}(g) = \{(x,y) \mapsto (x+u, y+v) | u, v \in \mathbf{K}\}.$$

We are still under the hypothesis that $\mathrm{char}(\mathbf{K}) = 0$ and G contains an element $f : (x,y) \mapsto (\alpha x, y+1)$ with $\alpha \in \mathbf{K}^*$. Assume now that no element of G has an infinite action on the base of the rational fibration $(x,y) \mapsto x$. Then the description of $\mathrm{Cent}(f)$ implies that G is a subgroup of

$$\{(x,y) \mapsto (\delta x, y+R(x)) | \delta \in \mathbf{K}^*, R \in \mathbf{K}(x)\}.$$

Consider the projection $\pi : G \to \mathrm{PGL}_2(\mathbf{K})$ which records the action on the base. Denote by G_0 the kernel of π and by G_b the image of π. We identify G_b as a subgroup of the multiplicative group of roots of unity of $\mathbf{K}$. We want to prove that G_b is finite so that G is virtually contained in $\mathrm{Jonq}_0(\mathbf{K}) = \mathrm{PGL}_2(\mathbf{K}(x))$. Assume that G_b is an infinite subgroup of the group of roots of unity. We first claim that G_0 is isomorphic to $\mathbf{K}$. Let $h : (x,y) \mapsto (x, y+R(x)), R \in \mathbf{K}(x)$ be an element of G_0 and $g : (x,y) \mapsto (\beta x, y+S(x)), S \in \mathbf{K}(x)$ be an element of G. The commutation relation $f \circ g = g \circ f$ implies $R(x) = R(\beta x)$. Here β can be any element of the infinite group G_b. This implies that R is constant, which proves the claim. Let H_γ be a finite subgroup of G_b, it is a cyclic group generated by $x \mapsto \gamma x$ for some $\gamma \in \mathbf{K}^*$. Let $g : (x,y) \mapsto (\gamma x, y+R(x))$ be an element of G such that $\pi(g)$ is $x \mapsto \gamma x$. By Lemma 2.11 R is a polynomial. We can conjugate g by an element of the form $(x,y) \mapsto (x, y+P(x)), P \in \mathbf{K}[x]$ to $(x,y) \mapsto (\gamma x, y)$ and the polynomial P is unique up to addition by a constant. In fact, the conjugation by $(x,y) \mapsto (x, y+P(x))$ sends the subgroup $\pi^{-1}(H_\gamma)$ of G into $\{(x,y)\mapsto(\delta x, y+t), t\in\mathbf{K}\}$ because any element h of $\pi^{-1}(H_\gamma)$ is equal to $g^n\circ g_0$ for some $n\in\mathbf{Z}$ and $g_0\in G_0$. The unicity of P implies that, if we take a finite subgroup H_ν which contains strictly H_γ, then the conjugation by $(x,y)\mapsto(x,y+P(x))$ still sends the subgroup $\pi^{-1}(H_\nu)$ into $\{(x,y)\mapsto(\delta x,y+t),t\in\mathbf{K}\}$. This further implies that the conjugation by $(x,y)\mapsto(x,y+P(x))$ sends the whole group G into $\{(x,y)\mapsto(\delta x,y+t),t\in\mathbf{K}\}$. Then by the maximality of G, it is isomorphic to $\mathbf{K}^*\times\mathbf{K}$ and we are in the first case of the statement. Note that we have made the hypothesis that G_b is torsion, so here $\mathbf{K}$ must be the algebraic closure of a finite field.

Assume that G contains an element $f:(x,y)\mapsto(\alpha x,\beta y)$ where $\alpha,\beta\in\mathbf{K}^*$ and β has infinite order. If α also has infinite order, then Theorem 2.7 implies immediately that $G=\mathrm{Cent}(f)$ is isomorphic to $\mathbf{K}^*\times\mathbf{K}^*$ and we are in the first case. Assume that α has finite order but G contains an element $f_1:(x,y)\dashrightarrow(\alpha_1 x, yR(x))$ where $R\in\mathbf{K}(x)$ and $\alpha_1\in\mathbf{K}^*$ has infinite order. By Corollary 3.11 the two elements f and f_1 are simultaneously conjugate to $(x,y)\mapsto(\alpha x,\beta y)$ and $(x,y)\mapsto(\alpha_1 x, ry)$ with $r\in\mathbf{K}^*$. Thus, Theorem 2.7, when applied respectively to f and f_1, shows that $G=\mathrm{Cent}(f)\bigcap\mathrm{Cent}(f_1)$ is isomorphic to the diagonal group $\mathbf{K}^*\times\mathbf{K}^*$. Hence we are in the first case.

According to the classification of normal forms of elliptic elements of infinite order (see Proposition 2.4), the only remaining cases are the two following: 1) G contains an element $f:(x,y)\mapsto(\alpha x,\beta y)$ where $\alpha\in\mathbf{K}^*$ has finite order and $\beta\in\mathbf{K}^*$ has infinite order but G contains no elements $(x,y)\dashrightarrow(\alpha_1 x,yR(x))$ with α_1 of infinite order; 2) $\mathrm{char}(\mathbf{K})=p>0$ and G contains an element $f:(x,y)\mapsto(x+1,\beta y)$ with $\beta\in\mathbf{K}^*$ of infinite order. In both cases $\mathrm{Cent}(f)$ is a subgroup of the Jonquières group by Theorem 2.7. Denote by π the projection of G into $\mathrm{PGL}_2(\mathbf{K})$. If $\pi(G)$ is finite then we are in the third case of Theorem 4.1. So we assume that $\pi(G)$ is infinite. Then $\pi(G)$ is isomorphic to an infinite subgroup of the group of roots of unity or an infinite subgroup of $\mathbf{K}$, and it is an infinite torsion abelian group. We want to show that we are in the second case of Theorem 4.1. By Lemma 2.9, each element of G is of the form $(x,y)\dashrightarrow(\eta(x),y\frac{rS(x)}{S(\eta(x))})$ with $\eta\in\mathrm{PGL}_2(\mathbf{K}), r\in\mathbf{K}^*, S\in\mathbf{K}(x)$. If $(x,y)\dashrightarrow(\eta(x),y\frac{rS(x)}{S(\eta(x))})$ is an element of G, then $(x,y)\dashrightarrow(\eta(x),y\frac{S(x)}{S(\eta(x))})$ is also an element of G because it commutes with every other element. However the later has the same order in G as η in $\mathrm{PGL}_2(\mathbf{K})$. This means that G has a subgroup isomorphic to $\pi(G)$, so that G is isomorphic to the product of this subgroup with the kernel of π. To finish the proof, it suffices to show that the kernel of π is $\{(x,y)\mapsto(x,\beta y)|\beta\in\mathbf{K}^*\}$. This is because $(x,y)\mapsto(x,\beta y)$ are the only possible elliptic elements by Lemma 2.9. □

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