

# The Cremona group of the plane is compactly presented

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## ABSTRACT

This article shows that the Cremona group of the plane is compactly presented. To do this, we prove that it is a generalized amalgamated product of three of its algebraic subgroups (automorphisms of the plane and Hirzebruch surfaces) divided by one relation.

## 1. Introduction

Let  $k$  be a field. The Cremona group  $\text{Bir}(\mathbb{P}^n)$  is the group of birational transformations of the projective space  $\mathbb{P}_k^n = \mathbb{P}^n$ . It corresponds to a very intensively studied topic in algebraic geometry (see [7, 10, 15] and references therein).

A birational transformation of  $\mathbb{P}^n$  is simply a birational change of coordinates, so  $\text{Bir}(\mathbb{P}^n)$  is a natural generalization of  $\text{Aut}(\mathbb{P}^n) = \text{PGL}_{n+1}(k)$ , and in many aspects the Cremona group behaves like semi-simple groups, but also in many aspects it does not. Some analogies between the Cremona groups and semi-simple groups have been presented by Serre in the 1000th Bourbaki seminar [15], and by Cantat [6].

For  $k$  a local field and endowed with the Euclidean topology, constructed in [5], the Cremona group becomes a Hausdorff topological group. For  $k = \mathbb{C}$  and  $k = \mathbb{R}$ , the restriction to its subgroup  $\text{PGL}_{n+1}(k)$  of linear coordinate changes of  $\mathbb{P}^n$  is the Euclidean topology. This not only opens the path to study the geometric properties of the Cremona group coming from the Euclidean topology, but also presents the opportunity to study the Cremona group from the point of view of geometric group theory and raises the question of analogies to Lie groups.

In this article, we will present one of these analogies, namely the property of being compactly presented (see Definition 6.1).

Let us take a closer look at the Cremona group endowed with the Euclidean topology:

The group  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^1) = \text{PGL}_2(\mathbb{C})$  is compactly presented by any neighbourhood of 1 because it is a connected complex algebraic group (see, for example, [1, Satz 3.1]).

For  $n \geq 2$  and  $k$  any local field, the group  $\text{Bir}(\mathbb{P}_k^n)$  is not locally compact [5, Lemma 5.15], though the topology is the inductive topology given by the family of closed sets  $\text{Bir}(\mathbb{P}^n)_{\leq d} = \{f \in \text{Bir}(\mathbb{P}^n) \mid \deg(f) \leq d\}$ , which are locally compact [5, Proposition 2.10, Lemma 5.4]. Furthermore, any compact subset of  $\text{Bir}(\mathbb{P}^n)$  is of bounded degree.

For  $n \geq 3$ , the group  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^n)$  is not compactly generated [5, Lemma 5.17], hence not compactly presented.

The group  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$  is generated by  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^2) = \text{PGL}_2(\mathbb{C})$  and the standard quadratic transformation  $\sigma : [x : y : z] \mapsto [yz : xz : xy]$  (see [8]). Its subgroup  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^2)$ , being a connected complex algebraic group, is compactly presented by any neighbourhood of 1. Hence  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$  is compactly generated by any compact neighbourhood of 1 in  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^2)$  and  $\sigma$ .

The aim of this article is to show that, even though it is neither an algebraic group nor locally compact,  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$  is moreover compactly presented.

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Received 19 November 2014; revised 29 June 2015; published online 12 November 2015.

2010 *Mathematics Subject Classification* 14E07, 20F05, 14L30 (primary).

The author gratefully acknowledges support by the Swiss National Science Foundation Grant ‘Birational geometry’ PP00P2\_153026/1.

**THEOREM A** (Corollary 6.8). *Endowed with the Euclidean topology, the Cremona group  $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$  is compactly presented by  $\{\sigma\} \cup K$ , where  $K$  is any compact neighbourhood of 1 in  $\text{Aut}(\mathbb{P}_{\mathbb{C}}^2)$  and  $\sigma : [x : y : z] \mapsto [yz : xz : xy]$  is the standard involution of  $\mathbb{P}_{\mathbb{C}}^2$ .*

For algebraically closed fields, the generating sets and generating relations of  $\text{Bir}(\mathbb{P}^2)$  have been studied thoroughly: The famous Noether–Castelnuovo theorem [8] states that if  $k$  is algebraically closed, then  $\text{Bir}(\mathbb{P}^2)$  is generated by  $\text{Aut}(\mathbb{P}^2)$  and the standard quadratic involution  $\sigma : [x : y : z] \mapsto [yz : xz : xy]$ , that is, the generating set is the union of two complex linear algebraic groups.

A presentation was given in [11], where the generating set consists of all quadratic transformations of  $\mathbb{P}^2$  and the generating relations are of the form  $q_1 q_2 q_3 = 1$ , where  $q_i$  are quadratic transformations. Another presentation was given in [14], where it is shown that  $\text{Bir}(\mathbb{P}^1 \times \mathbb{P}^1)$  (isomorphic to  $\text{Bir}(\mathbb{P}^2)$ ) is the amalgamated product of  $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$  and the de Jonquières group of birational maps of  $\mathbb{P}^1 \times \mathbb{P}^1$  preserving the first projection along their intersection modulo one relation. In [4], a similar result is presented; the group  $\text{Bir}(\mathbb{P}^2)$  is the amalgamated product of  $\text{Aut}(\mathbb{P}^2)$  and the de Jonquières group  $J_{[1:0:0]}$  of birational maps of  $\mathbb{P}^2$  preserving the pencil of lines through  $[1 : 0 : 0]$  along their intersection modulo one relation.

Since neither  $\text{Aut}(\mathbb{P}^2) = \text{PGL}_3(k)$  nor the set of quadratic transformations nor the de Jonquières group are compact in the Euclidean topology, these presentations yield no compact presentation. However, all three presentations yield bounded presentations (the length of the generating relations are universally bounded).

In [18], using [14], a presentation of  $\text{Bir}(\mathbb{P}^2)$  is given by the generalized amalgamated product of  $\text{Aut}(\mathbb{P}^2)$ ,  $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$  and  $J_{[0:1:0]}$  (as subgroups of  $\text{Bir}(\mathbb{P}^2)$ ) along their pairwise intersection, where  $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$  is viewed as a subgroup of  $\text{Bir}(\mathbb{P}^2)$  via a birational map  $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$  given by the pencils of lines through  $[0 : 1 : 0]$  and  $[1 : 0 : 0]$ . Again, since  $J_{[0:1:0]}$  is not compact, this does not yield a compact but only a bounded presentation, but it gives rise to the following idea, which is the key step in the proof of Theorem A.

**THEOREM B** (Theorem 5.5). *Let  $k$  be algebraically closed. Then the Cremona group  $\text{Bir}(\mathbb{P}^2)$  is isomorphic to the amalgamated product of  $\text{Aut}(\mathbb{P}^2)$ ,  $\text{Aut}(\mathbb{F}_2)$ ,  $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$  (as subgroups of  $\text{Bir}(\mathbb{P}^2)$ ) along their pairwise intersection in  $\text{Bir}(\mathbb{P}^2)$  modulo the relation  $\tau_{13}\sigma\tau_{13}\sigma$ , where  $\tau_{13} \in \text{Aut}(\mathbb{P}^2)$  is given by  $\tau_{13} : [x : y : z] \mapsto [z : y : x]$ .*

Here the inclusion of  $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$  into  $\text{Bir}(\mathbb{P}^2)$  is the same as before,  $\mathbb{F}_2$  is the second Hirzebruch surface and the inclusion of  $\text{Aut}(\mathbb{F}_2)$  into  $\text{Bir}(\mathbb{P}^2)$  is given by a birational map  $\mathbb{P}^2 \dashrightarrow \mathbb{F}_2$  given by the system of lines through  $[1 : 0 : 0]$ , and the point infinitely near corresponding to the tangent direction  $\{y = 0\}$ .

The method used to prove Theorem B is, like in [4, 14], to study linear systems and their base-points. The difference here is that our maps have bounded degree. This rigidifies the situation and changes the possibilities for simplifications. The proof of Theorem B does not use [18].

For  $k = \mathbb{C}$ , the three groups  $\text{Aut}(\mathbb{P}^2)$ ,  $\text{Aut}(\mathbb{F}_2)$ ,  $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$  are locally compact algebraic groups. Using this and Theorem B, we prove Theorem A.

The plan of the article is as follows:

In Sections 2 and 3, we give basic definitions and results on  $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$  and  $\text{Aut}(\mathbb{F}_2)$ . Section 4 is devoted to relations in the generalized amalgamated product of  $\text{Aut}(\mathbb{P}^2)$ ,  $\text{Aut}(\mathbb{F}_0)$ ,  $\text{Aut}(\mathbb{F}_2)$  modulo the relation  $\tau_{13}\sigma\tau_{13}\sigma$ . These are the backbone of the proof of Theorem B, which will be given in Section 5. In Section 6, we visit some facts about compactly presented groups and then finally prove Theorem A.

In Sections 2–5, we work over any algebraically closed field  $k$  and Section 6 restricts to  $k = \mathbb{C}$ .

## 2. Description of $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ and $\text{Aut}(\mathbb{F}_2)$ inside the Cremona group

This section is devoted to the description of the subgroups  $\text{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$  and  $\text{Aut}(\mathbb{F}_2)$  of  $\text{Bir}(\mathbb{P}^2)$ .

Remember that the  $n$ th Hirzebruch surface  $\mathbb{F}_n$ ,  $n \in \mathbb{N}$ , is given by

$$\mathbb{F}_n = \{([x : y : z], [u : v]) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid yv^n = zu^n\}.$$

Observe that  $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$  and that  $\mathbb{F}_1$  is isomorphic to the blow-up of one point in  $\mathbb{P}^2$ .

Consider the birational maps  $\varphi_0 : \mathbb{P}^2 \dashrightarrow \mathbb{F}_0$  and  $\varphi_2 : \mathbb{P}^2 \dashrightarrow \mathbb{F}_2$  given as follows: The map  $\varphi_0$  is given by the blow-up of the points  $[1 : 0 : 0]$  and  $[0 : 1 : 0]$  followed by the contraction of the line passing through them. The map  $\varphi_2$  is given by the blow-up of  $[1 : 0 : 0]$  and the point infinitely near  $[1 : 0 : 0]$  lying on the strict transform of  $\{y = 0\}$ , followed by the contraction of the strict transform of  $\{y = 0\}$ . The birational maps  $\varphi_0$  and  $\varphi_2$  are only defined up to automorphism of  $\mathbb{F}_0$  and  $\mathbb{F}_2$ . They induce homomorphisms of groups

$$\begin{aligned} \text{Aut}(\mathbb{F}_0) &\longrightarrow \text{Bir}(\mathbb{P}^2), & \psi &\longmapsto \varphi_0^{-1}\psi\varphi_0, \\ \text{Aut}(\mathbb{F}_2) &\longrightarrow \text{Bir}(\mathbb{P}^2), & \psi &\longmapsto \varphi_2^{-1}\psi\varphi_2 \end{aligned}$$

whose image is uniquely determined by the choice of points blown up in  $\mathbb{P}^2$ . We will denote the image of  $\text{Aut}(\mathbb{F}_i)$  also by  $\text{Aut}(\mathbb{F}_i)$  for  $i = 0, 2$  since no confusion occurs.

REMARK 2.1 (and Notation). (i) We can check that

$$\mathbb{P}^2 \dashrightarrow \mathbb{F}_0, \quad [x : y : z] \mapsto ([x : z], [y : z])$$

with inverse  $([u_0 : u_1], [v_0 : v_1]) \mapsto [u_0v_1 : v_0u_1 : u_1v_1]$ , and

$$\mathbb{P}^2 \dashrightarrow \mathbb{F}_2, \quad [x : y : z] \mapsto ([xy : y^2 : z^2], [y : z])$$

with inverse  $([u : v : w], [a : b]) \mapsto [ua : va : vb]$  are examples for  $\varphi_0$  and  $\varphi_2$ .

(ii) For  $i = 0, 2$ , the map  $(\varphi_i)^{-1}$  has exactly one base-point, which we denote by  $p_i$  (Figure 1).

(iii) The image of the linear system of lines of  $\mathbb{P}^2$  by  $\varphi_i$  has a unique base-point, namely  $p_i$ .

(iv) We denote by  $C_1$  the curve of self-intersection 0 in  $\mathbb{F}_0$  which is contracted by  $(\varphi_0)^{-1}$  onto  $[1 : 0 : 0]$ , and by  $C_2$  the curve of self-intersection 0 which is contracted onto  $[0 : 1 : 0]$ . Remark that  $p_0 = \varphi_0(\{z = 0\})$  and that  $\{p_0\} = C_1 \cap C_2$ .

(v) We denote by  $E$  the exceptional curve of self-intersection  $-2$  in  $\mathbb{F}_2$ . It is contracted onto  $[1 : 0 : 0]$  by  $(\varphi_2)^{-1}$ . Denote by  $C$  the curve of self-intersection 0 in  $\mathbb{F}_2$  which is contracted by  $(\varphi_2)^{-1}$  onto the point infinitely near  $[1 : 0 : 0]$  corresponding to the tangent  $\{y = 0\}$ . Remark that  $p_2 = \varphi_2(\{y = 0\})$  and  $p_2 \in C \setminus E$ .

(vi) Let  $L \subset \mathbb{P}^2$  be a general line. Then  $C_j \cdot \varphi_0(L) = 1$ ,  $j = 1, 2$ , and  $C \cdot \varphi_2(L) = 1$ ,  $E \cdot \varphi_2(L) = 0$ .

The following picture illustrates for  $i = 0, 2$  the transformation  $(\varphi_i)^{-1}\psi_i\varphi_i$ , where  $\psi_i$  is some automorphism of  $\mathbb{F}_i$ . At the same time it shows the blow-up diagram of  $(\varphi_i)^{-1}\psi_i\varphi_i$ .

Consider the birational transformations of  $\mathbb{P}^2$  given by

$$\begin{aligned} \sigma_1 : [x : y : z] &\mapsto [-xy + z^2 : y^2 : yz], \\ \sigma_2 : [x : y : z] &\mapsto [xy : z^2 : yz], \\ \sigma_3 : [x : y : z] &\mapsto [yz : xz : xy]. \end{aligned}$$

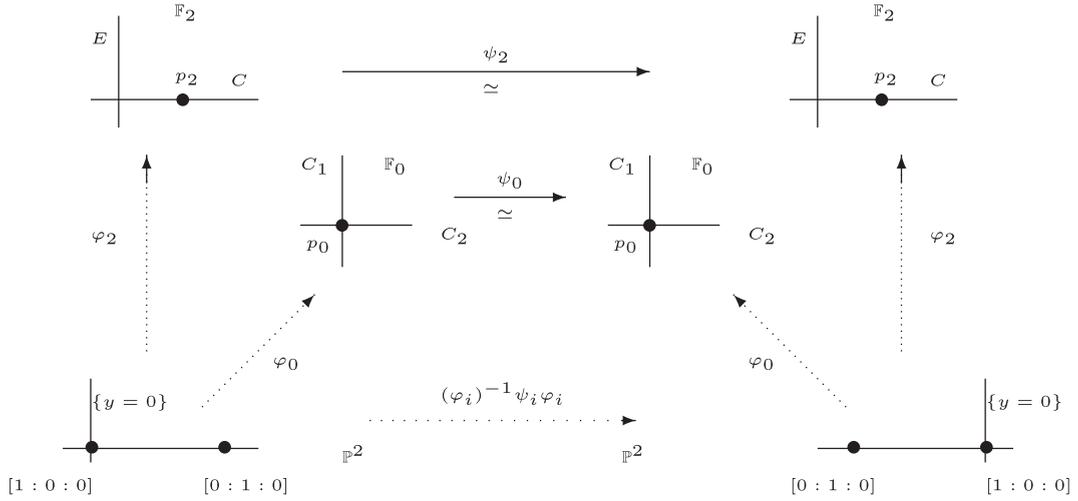


FIGURE 1. The transformation  $(\varphi_i)^{-1} \psi_i \varphi_i$  for  $i = 0, 2$ .

They are three quadratic involutions of  $\mathbb{P}^2$  with, respectively, exactly one, two and three proper base-points in  $\mathbb{P}^2$ . The map  $\sigma_3$  is usually referred to as standard quadratic involution of  $\mathbb{P}^2$ .

The map  $\sigma_3$  has base-points  $[1:0:0], [0:1:0], [0:0:1]$ , the map  $\sigma_2$  has base-points  $[1:0:0], [0:1:0]$  and the point  $p$  infinitely near  $[1:0:0]$  corresponding to the direction  $\{y=0\}$ , and the map  $\sigma_1$  has base-points  $[1:0:0], p, q$ , where  $q$  is a point infinitely near  $p$  not contained in the intersection of the strict transform of the exceptional divisor of  $[1:0:0]$ .

**REMARK 2.2.** For any quadratic map  $\tau \in \text{Bir}(\mathbb{P}^2)$  we can find  $i = 1, 2, 3$  and  $\alpha, \beta \in \text{Aut}(\mathbb{P}^2)$  such that  $\alpha$  sends the base-points of  $\tau$  onto the base-points of  $\sigma_i$  and  $\beta$  sends the base-points of  $\tau^{-1}$  onto the base-points of  $\sigma_i$ . We can then write  $\tau = \beta^{-1} \sigma_i \alpha$  (see [2, Subsections 2.1 and 2.8]).

It follows that the linear system of  $\tau$  is the image of the linear system of  $\sigma_i$  by  $\alpha^{-1}$ , and that  $\tau$  and  $\sigma_i$  have the same amount of proper base-points in  $\mathbb{P}^2$ . Since  $\sigma_1, \sigma_2, \sigma_3$  have, respectively, one, two and three proper base-points in  $\mathbb{P}^2$ , the amount of proper base-points of  $\tau$  determines  $i$ .

The following is the description of the groups  $\text{Aut}(\mathbb{F}_0)$  and  $\text{Aut}(\mathbb{F}_2)$  as subgroups of  $\text{Bir}(\mathbb{P}^2)$  given by the above inclusions:

**LEMMA 2.3.** (i) For  $i = 0, 2$ , the group  $\mathcal{A}_i := \text{Aut}(\mathbb{F}_i) \cap \text{Aut}(\mathbb{P}^2)$  is the group of automorphisms of  $\mathbb{P}^2$  fixing the set of base-points of  $\varphi_i$ , that is, the set  $\{[1:0:0], [0:1:0]\}$  if  $i = 0$ , and the point  $[1:0:0]$  and the line  $\{y=0\}$  if  $i = 2$ .

For each  $i \in \{0, 2\}$ ,  $\mathcal{A}_i$  corresponds via  $\varphi_i$  to the set of automorphisms of  $\mathbb{F}_i$  that fix  $p_i$ .

(ii) The set  $\text{Aut}(\mathbb{F}_0) \setminus \text{Aut}(\mathbb{P}^2)$  consists of all elements of the form  $\beta \sigma_i \alpha$ , where  $i = 2, 3$  and  $\alpha, \beta \in \text{Aut}(\mathbb{F}_0) \cap \text{Aut}(\mathbb{P}^2)$ .

(iii) The set  $\mathcal{A}_0 \cup \mathcal{A}_0 \sigma_2 \mathcal{A}_0$  corresponds via  $\varphi_0$  to the set of automorphisms of  $\mathbb{F}_0$ , sending  $p_0$  into  $C_1 \cup C_2$ .

(iv) The set  $\text{Aut}(\mathbb{F}_2) \setminus \text{Aut}(\mathbb{P}^2)$  consists of all elements of the form  $\beta \sigma_i \alpha$ , where  $i = 1, 2$  and  $\alpha, \beta \in \text{Aut}(\mathbb{F}_2) \cap \text{Aut}(\mathbb{P}^2)$ .

(v) The set  $\mathcal{A}_2 \cup \mathcal{A}_2 \sigma_1 \mathcal{A}_2$  corresponds via  $\varphi_2$  to the set of automorphisms of  $\mathbb{F}_2$  that send  $p_2$  into  $C$ .



LEMMA 2.4. For  $i = 0, 2$ , let  $\mathcal{A}_i := \text{Aut}(\mathbb{P}^2) \cap \text{Aut}(\mathbb{F}_i)$ .

- (i) The groups  $\text{Aut}(\mathbb{F}_0)$  and  $\text{Aut}(\mathbb{F}_2)$  are linear algebraic subgroups of  $\text{Bir}(\mathbb{P}^2)$ .
- (ii) The group  $\text{Aut}(\mathbb{F}_0)$  has two irreducible components, namely the component  $\text{Aut}(\mathbb{F}_0)^0$  containing 1 and  $\tau_{12}\text{Aut}(\mathbb{F}_0)^0$ , where  $\tau_{12} \in \text{Aut}(\mathbb{F}_0)$  is given by  $\tau_{12}: [x : y : z] \mapsto [y : x : z]$ .
- (iii) The group  $\text{Aut}(\mathbb{F}_2)$  is irreducible.
- (iv) The set  $\mathcal{A}_0\sigma_3\mathcal{A}_0$  is a Zariski-open set of  $\text{Aut}(\mathbb{F}_0)$ .
- (v) The set  $\mathcal{A}_2\sigma_2\mathcal{A}_2$  is a Zariski-open set of  $\text{Aut}(\mathbb{F}_2)$ .

*Proof.* (i)–(iii) are classical results, which, for example, can be found in [3, Proposition 2.2.6, Théorème 2].

(iv) By Lemma 2.3, the set  $\text{Aut}(\mathbb{F}_0) \setminus (\mathcal{A}_0\sigma_3\mathcal{A}_0)$  is the set of elements of  $\text{Aut}(\mathbb{F}_0)$  that send the point  $p_0$  into the curve  $C_1 \cup C_2$  and is therefore closed.

(v) By Lemma 2.3, the set  $\text{Aut}(\mathbb{F}_2) \setminus (\mathcal{A}_2\sigma_2\mathcal{A}_2)$  is the set of elements of  $\text{Aut}(\mathbb{F}_2)$  that fix the curve  $C$  and is therefore closed.  $\square$

REMARK 2.5. The Noether–Castelnuovo theorem states that  $\text{Bir}(\mathbb{P}^2)$  is generated by  $\text{Aut}(\mathbb{P}^2)$  and  $\sigma_3$  ([8], see also [2, Section 8]). Furthermore, we can write  $\sigma_3 = \tau_{12}\sigma_2\tau_{12}\sigma_2$  where  $\tau_{12}([x : y : z]) = ([y : x : z])$ , hence the group  $\text{Bir}(\mathbb{P}^2)$  is also generated by  $\text{Aut}(\mathbb{P}^2)$  and  $\sigma_2$ . Therefore, for any  $i = 0, 2$ , the group  $\text{Bir}(\mathbb{P}^2)$  is generated by its subgroups  $\text{Aut}(\mathbb{P}^2)$  and  $\text{Aut}(\mathbb{F}_i)$ , and thus also generated by all three subgroups  $\text{Aut}(\mathbb{P}^2)$ ,  $\text{Aut}(\mathbb{F}_0)$  and  $\text{Aut}(\mathbb{F}_2)$ .

Note that Lemma 2.3 in particular implies that all elements of  $\text{Aut}(\mathbb{F}_0) \cup \text{Aut}(\mathbb{F}_2)$  are linear or quadratic.

DEFINITION 2.6. For a set  $S$ , let  $F_S$  be the free group generated by  $S$ .

For the set  $S := \text{Aut}(\mathbb{P}^2) \cup \text{Aut}(\mathbb{F}_0) \cup \text{Aut}(\mathbb{F}_2) \subset \text{Bir}(\mathbb{P}^2)$ , define

$$\mathfrak{G} := F_S \left/ \left\langle \begin{array}{l} fgh^{-1}, \quad \text{if } fg = h \text{ in } \text{Aut}(\mathbb{P}^2) \\ fgh^{-1}, \quad \text{if } fg = h \text{ in } \text{Aut}(\mathbb{F}_0) \\ fgh^{-1}, \quad \text{if } fg = h \text{ in } \text{Aut}(\mathbb{F}_2) \\ \tau_{13}\sigma_3\tau_{13}\sigma_3 \end{array} \right\rangle \right.$$

where  $\tau_{13} \in \text{Aut}(\mathbb{P}^2)$  is given by  $\tau_{13}: [x : y : z] \mapsto [z : y : x]$ .

REMARK 2.7. The group  $\mathfrak{G}$  is isomorphic to the free product of the three groups  $\text{Aut}(\mathbb{P}^2)$ ,  $\text{Aut}(\mathbb{F}_0)$ ,  $\text{Aut}(\mathbb{F}_2)$  amalgamated along all the pairwise intersections (generalized amalgamated product of the three groups) modulo the relation  $\tau_{13}\sigma_3 = \sigma_3\tau_{13}$ .

A geometric approach to generalized amalgamated products can be found in [13, 16, 17]. The generalized amalgamated product

$$F_S \left/ \left\langle \begin{array}{l} fgh^{-1}, \quad \text{if } fg = h \text{ in } \text{Aut}(\mathbb{P}^2) \\ fgh^{-1}, \quad \text{if } fg = h \text{ in } \text{Aut}(\mathbb{F}_0) \\ fgh^{-1}, \quad \text{if } fg = h \text{ in } \text{Aut}(\mathbb{F}_2) \end{array} \right\rangle \right.$$

is in [17, Subsection 1.3] the colimit of the diagram

$$\begin{array}{ccccc} & & \text{Aut}(\mathbb{P}^2) & & \\ & \swarrow & & \searrow & \\ \mathcal{A}_0 & \text{---} & \mathcal{A}_0 \cap \mathcal{A}_2 & \text{---} & \mathcal{A}_2 \\ & \swarrow & & \searrow & \\ \text{Aut}(\mathbb{F}_0) & \text{---} & \text{Aut}(\mathbb{F}_0) \cap \text{Aut}(\mathbb{F}_2) & \text{---} & \text{Aut}(\mathbb{F}_2) \end{array}$$

Equivalently, it is the fundamental group of a 2-complex of groups. The vertices are  $\text{Aut}(\mathbb{P}^2)$ ,  $\text{Aut}(\mathbb{F}_0)$ ,  $\text{Aut}(\mathbb{F}_2)$ , the edges are their pairwise intersection and the 2-simplex is the group  $\text{Aut}(\mathbb{P}^2) \cap \text{Aut}(\mathbb{F}_0) \cap \text{Aut}(\mathbb{F}_2)$  [13, Subsections 2.1 and 3.4, 16, 4.4].

REMARK 2.8. By Remark 2.5, there exists a canonical surjective homomorphism of groups

$$\pi : \mathfrak{G} \longrightarrow \text{Bir}(\mathbb{P}^2)$$

and by definition of  $\mathfrak{G}$  a natural map

$$w : \text{Aut}(\mathbb{P}^2) \cup \text{Aut}(\mathbb{F}_0) \cup \text{Aut}(\mathbb{F}_2) \longrightarrow \mathfrak{G}$$

which sends an element to its corresponding word. Note that  $\pi \circ w$  is the identity map.

### 3. Base-points, multiplicities, de Jonquières

The methods we use mainly consist of studying linear systems of  $\mathbb{P}^2$  and their base-points. In this section, we recall some definitions, notions and formulae which will be used almost constantly in Section 4 and 5, which have the aim to prove Theorem B (Theorem 5.5).

DEFINITION 3.1. A *point over  $\mathbb{P}^2$*  is a point  $p \in S$ , where  $S := S_{n+1} \xrightarrow{\nu_n} S_{n-1} \xrightarrow{\nu_{n-1}} \cdots \xrightarrow{\nu_1} S_0 := \mathbb{P}^2$  is a sequence of blow-ups, and where we identify  $p \in S$  with  $p_i \in S_i$  if  $\nu_{i+1} \cdots \nu_n : S \rightarrow S_i$  is a local isomorphism around  $p$  sending  $p$  to  $p_i$ .

A point  $p \in S$  over  $\mathbb{P}^2$  is *proper* if it is equivalent to a point  $p' \in \mathbb{P}^2$ , and *infinitely near* otherwise.

DEFINITION 3.2. Let  $f \in \text{Bir}(\mathbb{P}^2)$  be a quadratic birational transformation, and call  $p_1, p_2, p_3$  its base-points and  $q_1, q_2, q_3$  the base-points of  $f^{-1}$ . We say that *the base-points of  $f$  are ordered consistently* if the following holds: The base-points of  $f$  and of  $f^{-1}$  are ordered such that the following conditions are satisfied.

(i) If  $p_1, p_2, p_3$  are proper points of  $\mathbb{P}^2$ , then all the lines through  $p_1$  (respectively,  $p_2, p_3$ ) are sent onto lines through  $q_1$  (respectively,  $q_2, q_3$ ).

(ii) If  $p_1, p_2$  are proper points of  $\mathbb{P}^2$  and  $p_3$  is infinitely near to  $p_1$ , then all the lines through  $p_1$  (respectively,  $p_2$ ) are sent onto lines through  $q_1$  (respectively,  $q_2$ ).

(iii) If  $p_1$  is a proper point of  $\mathbb{P}^2$ ,  $p_2$  infinitely near  $p_1$  and  $p_3$  infinitely near  $p_2$ , then the lines through  $p_1$  are sent onto lines through  $q_1$  and the exceptional curve associated to  $p_3$  is sent onto the tangent associated to  $q_2$ .

REMARK 3.3. Writing down the blow-up diagram of the three quadratic involutions  $\sigma_1, \sigma_2, \sigma_3$ , we see that we can always order their base-points consistently (for example, for  $\sigma_3$  the ordering  $p_1 = q_1 = [1 : 0 : 0], p_2 = q_2 = [0 : 1 : 0], p_3 = q_3 = [0 : 0 : 1]$  is consistent). Since any quadratic birational transformation of  $\mathbb{P}^2$  can be written  $\beta\sigma_i\alpha$  for some suitable  $i \in \{1, 2, 3\}$ ,  $\alpha, \beta \in \text{Aut}(\mathbb{P}^2)$  (Remark 2.2), it is always possible to order its base-points consistently.

Throughout the article, we will always assume that the base-points of a quadratic transformation of  $\mathbb{P}^2$  are ordered consistently.

Let us remind the reader of the following formula: Let  $\Delta$  be a linear system and  $f \in \text{Bir}(\mathbb{P}^2)$  be a quadratic transformation with base-points  $p_1, p_2, p_3$ , and  $q_1, q_2, q_3$  the base-points of  $f^{-1}$ . Let  $a_i$  be the multiplicity of  $\Delta$  in  $p_i$  and  $b_i$  be the multiplicity of  $f(\Delta)$  in  $q_i$ . If the base-points of  $f$  are ordered consistently, then

$$\deg(f(\Delta)) = 2 \deg(\Delta) - \varepsilon, \quad b_i = \deg(\Delta) - \varepsilon + a_i$$

for  $i = 1, 2, 3$  and  $\varepsilon = a_1 + a_2 + a_3$  (see [2, Subsection 4.2]).

DEFINITION 3.4. We define

$$J := \{f \in \text{Bir}(\mathbb{P}^2) : f \text{ preserves the pencil of lines through } [1 : 0 : 0]\}.$$

The elements of  $J$  are called *de Jonquière transformations*.

A linear system  $\Delta$  of  $\mathbb{P}^2$  of degree  $\deg(\Delta) = d$  and with base-points  $p_1, \dots, p_n$  of multiplicity  $a_1, \dots, a_n$  is called *de Jonquière linear system* if it has multiplicity  $d - 1$  at  $[1 : 0 : 0]$  and satisfies the conditions  $d^2 - 1 = \sum_{i=1}^n a_i^2$  and  $3(d - 1) = \sum_{i=1}^n a_i$ .

We call a base-point of  $f$  a *simple base-point* if it is different from  $[1 : 0 : 0]$ , and denote the set of simple base-points by  $\text{sBp}(f)$ .

REMARK 3.5. (1) We have the following inclusions:  $\text{Aut}(\mathbb{F}_2) \subset J$  and  $\text{Aut}(\mathbb{F}_0)^0 \subset J$ , where  $\text{Aut}(\mathbb{F}_0)^0$  is the connected component of  $\text{Aut}(\mathbb{F}_0)$  containing  $\text{Id}$  and which is equal to  $(\mathcal{A}\sigma_2\mathcal{A}) \cup (\mathcal{A}\tau_{12}\sigma_2\tau_{12}\mathcal{A}) \cup (\mathcal{A}\sigma_3\mathcal{A}) \cup \mathcal{A}$ , where  $\mathcal{A} = \{\alpha \in \text{Aut}(\mathbb{P}^2) \cap \text{Aut}(\mathbb{F}_2) \mid \alpha([1 : 0 : 0]) = [1 : 0 : 0]\}$  and  $\tau_{12} \in \text{Aut}(\mathbb{P}^2)$  is given by  $\tau_{12} : [x : y : z] \mapsto [y : x : z]$  (Lemma 2.4).

(2) Any element of  $f \in J \setminus \text{Aut}(\mathbb{P}^2)$  of degree  $d$  has  $2d - 1$  base-points: the base-point  $[1 : 0 : 0]$  of multiplicity  $d - 1$  and  $2d - 2$  other base-points of multiplicity one (this follows from the conditions on the degree and multiplicities). Thus the definition of simple base-point of  $f$  is quite natural. If  $f \in J$  is of degree 2, then it has exactly three base-points, all of multiplicity one. Its simple base-points are just the ones different from  $[1 : 0 : 0]$ .

(3) A de Jonquière linear system of  $\mathbb{P}^2$  of degree  $d$  has  $2d - 1$  base-points and the multiplicity at any base-point different from  $[1 : 0 : 0]$  is one. Such a point is called a *simple base-point* of  $\Delta$ . Observe that, for  $f \in J$  and  $\Delta$  a de Jonquière linear system,  $f(\Delta)$  is a de Jonquière linear system, and the linear system of  $f$  is a de Jonquière linear system.

LEMMA 3.6. For any quadratic de Jonquière transformation  $f \in \text{Bir}(\mathbb{P}^2)$  there exist  $\alpha_1, \alpha_2 \in \text{Aut}(\mathbb{P}^2) \cap J$ ,  $\tau \in \{\sigma_1, \sigma_2, \tau_{12}\sigma_2\tau_{12}, \sigma_3\} \subset \text{Aut}(\mathbb{F}_0) \cup \text{Aut}(\mathbb{F}_2)$ , where  $\tau_{12} \in \text{Aut}(\mathbb{P}^2)$  is given by  $\tau_{12} : [x : y : z] \mapsto [y : x : z]$ , such that  $f = \alpha_2\tau\alpha_1$ .

In particular,  $(\text{Aut}(\mathbb{P}^2) \cup \text{Aut}(\mathbb{F}_0) \cup \text{Aut}(\mathbb{F}_2)) \cap \mathcal{J}$  generates  $\mathcal{J}$ .

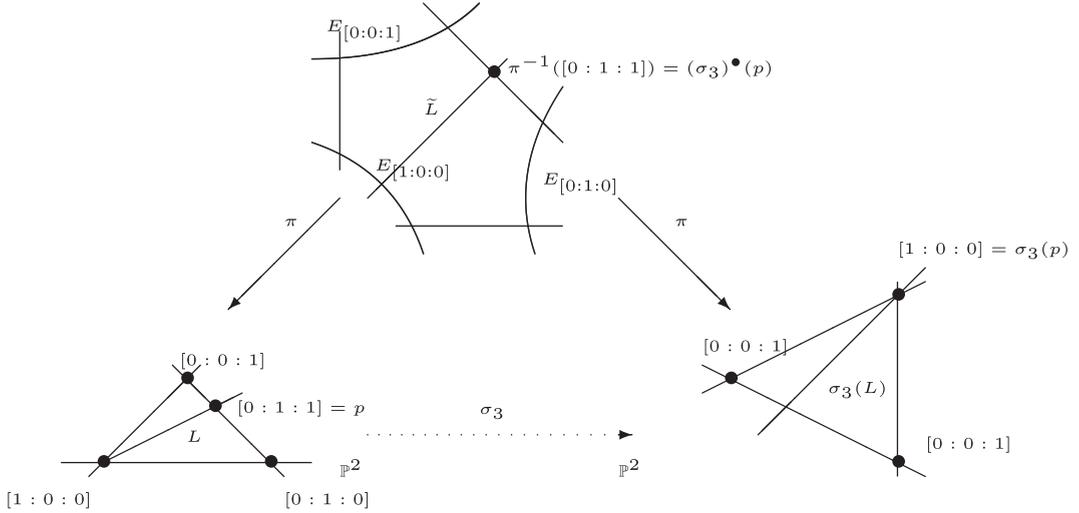
*Proof.* By Remark 2.1, we can write  $f = \alpha_2\sigma_i\alpha_1$  for some  $\alpha_1, \alpha_2 \in \text{Bir}(\mathbb{P}^2)$  and  $i$  is determined by the amount of proper base-points of  $f$ . Since  $f$  is de Jonquière, the point  $[1 : 0 : 0]$  is a base-point of  $f$ .

If  $f$  has only one proper base-point in  $\mathbb{P}^2$ , then it has to be fixed by  $\alpha_1$  and  $\alpha_2$ , which belong thus to  $J$ . This gives the result.

Suppose that  $f$  has exactly two proper base-points, namely  $[1 : 0 : 0]$  and  $p$ . This implies that  $\sigma_i = \sigma_2$ , which has base-points  $[1 : 0 : 0], [0 : 1 : 0]$  and a third one, infinitely near  $[1 : 0 : 0]$ . The base-point of  $f$  which is not a proper point of  $\mathbb{P}^2$  is either infinitely near  $[1 : 0 : 0]$  or  $p$ . If it is infinitely near  $[1 : 0 : 0]$ , then  $\alpha_1, \alpha_2$  fix  $[1 : 0 : 0]$  and are therefore de Jonquière. If it is infinitely near  $p$ , then  $\alpha_1$  sends  $p$  onto  $[1 : 0 : 0]$  and  $[1 : 0 : 0]$  onto  $[0 : 1 : 0]$ . We write  $\alpha_1 = \tau_{12}\beta_1$ ,  $\alpha_2 = \beta_2\tau_{12}$ , for some  $\beta_1, \beta_2 \in \text{Aut}(\mathbb{P}^2)$ , which means that  $\beta_1$  fixes  $[1 : 0 : 0]$ , that is,  $\beta_1 \in J \cap \text{Aut}(\mathbb{P}^2)$  and  $f = \beta_2(\tau_{12}\sigma_2\tau_{12})\beta_1$ . Since  $f, \beta_1, \tau_{12}\sigma_2\tau_{12} \in J$ , we have  $\beta_2 \in J$ .

Suppose that  $f$  has three proper base-points. For any  $\theta \in \text{Aut}(\mathbb{P}^2)$  that permutes the coordinate  $x, y, z$ , we have  $\theta\sigma_3\theta = \sigma_3$ . Therefore, we can assume that  $\alpha_1 \in J$ . Since  $\sigma_3$  is de Jonquière, the map  $\alpha_2$  has to be de Jonquière as well.

Since every element of  $\mathcal{J}$  decomposes into quadratic elements of  $\mathcal{J}$  (see [2, Theorem 8.4.3]), Lemma 3.6 implies that  $\mathcal{J}$  is generated by  $(\text{Aut}(\mathbb{P}^2) \cup \text{Aut}(\mathbb{F}_0) \cup \text{Aut}(\mathbb{F}_2)) \cap \mathcal{J}$  generates  $\mathcal{J}$ .  $\square$


 FIGURE 2. The points  $(\sigma_3)^\bullet(p)$  and  $\sigma_3(p)$ .

REMARK 3.7. Suppose  $\Delta$  is a de Jonquières linear system of degree  $d$  and  $f$  a quadratic de Jonquières transformation. We can say the following about the degree of  $f(\Delta)$ : Let  $p_1 = [1 : 0 : 0]$ ,  $p_2, p_3$  be the base-points of  $f$  and  $a_i$  be the multiplicity of  $\Delta$  in  $p_i$ . Then  $a_1 = \deg(\Delta) - 1$  and by the formula given above, we have

$$\deg(f(\Delta)) = 2d - (d - 1) - a_2 - a_3 = d + 1 - a_2 - a_3.$$

Since  $\Delta$  has one base-point of multiplicity  $d - 1$  and all the other base-points are of multiplicity 1, we know that, for  $i = 2, 3$ ,  $a_i$  is either zero or one. In fact,  $a_i = 0$  if  $p_i$  is not a common base-point of  $f$  and  $\Delta$ , and  $a_i = 1$  if  $p_i$  is a common base-point of  $f$  and  $\Delta$ . Thus the formula implies

$$\deg(f(\Delta)) = \begin{cases} d + 1 & \text{if } f \text{ and } \Delta \text{ have no common simple base-points,} \\ d & \text{if } f \text{ and } \Delta \text{ have exactly one common simple base-point,} \\ d - 1 & \text{if } f \text{ and } \Delta \text{ have exactly two common simple base-points.} \end{cases}$$

Furthermore, if  $p$  is a simple base-point of  $\Delta$  that is not a base-point of  $f$ , then  $f^\bullet(p)$  (see definition below) is a simple base-point of  $f(\Delta)$  [2, Subsection 4.1].

DEFINITION 3.8. Let  $f \in \text{Bir}(\mathbb{P}^2)$  and  $p$  be a point over the domain  $\mathbb{P}^2$  that is not a base-point of  $f$ . Take a minimal resolution of  $f$

$$\begin{array}{ccc} & S & \\ \nu_1 \swarrow & & \searrow \nu_2 \\ \mathbb{P}^2 & \xrightarrow{f} & \mathbb{P}^2 \end{array}$$

where  $\nu_1, \nu_2$  are sequences of blow-ups. Let  $p' \in S$  be a representative of  $p$ . We can see  $p'$  as a point over the range  $\mathbb{P}^2$ , and call it  $f^\bullet(p)$ .

Let us look at an example to understand  $f^\bullet(p)$  and  $f(p)$ :

EXAMPLE 3.9. Consider the standard quadratic involution  $\sigma_3 \in \text{Bir}(\mathbb{P}^2)$  and the point  $p = [0 : 1 : 1]$ , which is on the line  $\{x = 0\}$  contracted by  $\sigma_3$  onto the point  $[1 : 0 : 0]$ , which

means that  $\sigma_3(p) = [1 : 0 : 0]$ . The line  $L = \{y = z\}$  passing through  $p$  and  $[1 : 0 : 0]$  is sent by  $\sigma_3$  onto itself. By definition,  $(\sigma_3)^\bullet(p)$  is the point in the first neighbourhood of  $[1 : 0 : 0]$  corresponding to the tangent direction  $\{y = z\}$ . In conclusion,  $\sigma_3(p)$  is a proper point of  $\mathbb{P}^2$ , whereas  $(\sigma_3)^\bullet(p)$  is not. The following picture (Figure 2) illustrates the situation.

REMARK 3.10. Note that  $f^\bullet$  is a one-to-one correspondence between the sets

$$\begin{aligned} &(\mathbb{P}^2 \cup \{\text{infinitely near points}\}) \setminus \{\text{base-points of } f\} \quad \text{and} \\ &(\mathbb{P}^2 \cup \{\text{infinitely near points}\}) \setminus \{\text{base-points of } f^{-1}\}. \end{aligned}$$

#### 4. Basic relations in $\mathfrak{G}$

In this section, we present basic relations that hold in  $\mathfrak{G}$  and which will be the backbone of the proof of Theorem A (Theorem 5.5). We prove relations for words in  $\mathfrak{G}$  of length three using properties of the elements of  $\text{Aut}(\mathbb{F}_0)$ ,  $\text{Aut}(\mathbb{F}_2)$  and  $\text{Aut}(\mathbb{P}^2)$ . (Lemmas 4.2–4.4.) They will then be used in the next section to prove that there exists an injective map  $w_J: J \rightarrow \mathfrak{G}$  such that  $\pi \circ w_J = \text{Id}$  (Lemma 5.1, Corollary 5.2), which will enable us to prove Theorem A (Theorem 5.5) using the result that  $\text{Bir}(\mathbb{P}^2)$  is the amalgamated product of  $\text{Aut}(\mathbb{P}^2)$  and  $J$  modulo one relation [4] (Theorem 5.3).

Lemmas 4.1 and 4.2 yield that words of length three in  $\mathfrak{G}$  whose image in  $\text{Bir}(\mathbb{P}^2)$  is linear or quadratic behave like their images in  $\text{Bir}(\mathbb{P}^2)$ . Lemmas 4.3 and 4.4 yield relations for words of length three whose image in  $\text{Bir}(\mathbb{P}^2)$  is de Jonquières and of degree 3.

Define  $\text{TAut}(\mathbb{P}^2) = D \rtimes S_3$ , where  $S_3 \subset \text{Aut}(\mathbb{P}^2)$  is the image of the permutation matrices of  $\text{GL}_3$  and  $D$  is the image of the three-dimensional torus. We can check that the group  $\text{TAut}(\mathbb{P}^2)$  is normalized by  $\sigma_3$ , and the automorphism of  $\text{TAut}(\mathbb{P}^2)$  given by the conjugation of  $\sigma_3$  will be denoted by  $\iota$ . Note that  $\iota(\alpha) = \alpha$  for  $\alpha \in S_3$  and  $\iota(\delta) = \delta^{-1}$  for  $\delta \in D$ .

As subgroup of  $\text{Aut}(\mathbb{P}^2)$ , we can embed  $\text{TAut}(\mathbb{P}^2)$  (as a set) into  $\mathfrak{G}$  by the word map  $w$ . Lemma 4.1 shows that in  $\mathfrak{G}$  the image of  $\text{TAut}(\mathbb{P}^2)$  is normalized by  $w(\sigma_3)$ :

LEMMA 4.1. *For any  $(\delta, \alpha) \in D \rtimes S_3$  the relation  $w(\delta\alpha)w(\sigma_3) = w(\sigma_3)w(\iota(\delta\alpha))$  holds in  $\mathfrak{G}$ .*

*Proof.* Let  $\tau_{12}: [x : y : z] \mapsto [y : x : z]$ . In  $\text{Aut}(\mathbb{F}_0)$ , the relation  $\tau_{12}\sigma_3\tau_{12} = \sigma_3$  holds, hence the relation  $w(\tau_{12})w(\sigma_3) = w(\sigma_3)w(\tau_{12})$  holds in  $\mathfrak{G}$ . By definition,  $w(\tau_{13})w(\sigma_3) = w(\sigma_3)w(\tau_{13})$  is a relation in  $\mathfrak{G}$ , and  $\tau_{13}$  and  $\tau_{12}$  generate  $S_3$ . Therefore, the relation  $w(\alpha)w(\sigma_3) = w(\sigma_3)w(\alpha) = w(\sigma_3)w(\iota(\alpha))$  holds in  $\mathfrak{G}$  for any  $\alpha \in S_3$ .

Let  $\delta \in D$ . The relation  $\delta\sigma_3\delta = \sigma_3$  holds in  $\text{Aut}(\mathbb{F}_0)$ , hence  $w(\delta)w(\sigma_3) = w(\sigma_3)w(\delta^{-1}) = w(\sigma_3)w(\iota(\delta))$  holds in  $\mathfrak{G}$ .  $\square$

Using Lemma 4.1, we now show that, for  $f, g, h \in \text{Aut}(\mathbb{P}^2) \cup \text{Aut}(\mathbb{F}_0) \cup \text{Aut}(\mathbb{F}_2)$  and  $\deg(fgh) \leq 2$ , the word  $w(f)w(g)w(h)$  behaves like the composition  $fgh$ .

LEMMA 4.2. *Let  $g \in \text{Aut}(\mathbb{P}^2)$ ,  $h, f \in \text{Aut}(\mathbb{F}_0) \cup \text{Aut}(\mathbb{F}_2)$  such that  $\deg(fgh) \in \{1, 2\}$ .*

- (i) *If  $\deg(fgh) = 1$ , then  $w(f)w(g)w(h) = w(fgh)$  in  $\mathfrak{G}$ .*
- (ii) *If  $\deg(fgh) = 2$ , then there exist  $\alpha, \beta \in \text{Aut}(\mathbb{P}^2)$ ,  $\tilde{g} \in \text{Aut}(\mathbb{F}_0) \cup \text{Aut}(\mathbb{F}_2)$  such that  $w(f)w(g)w(h) = w(\beta)w(\tilde{g})w(\alpha)$  in  $\mathfrak{G}$ .*
- (iii) *If  $\deg(fgh) = 2$  and  $f, g, h \in J$ , then  $\alpha, \beta, \tilde{g}$  can be chosen to be in  $J$ .*

*Proof.* Suppose that  $f \in \text{Aut}(\mathbb{P}^2)$  or  $h \in \text{Aut}(\mathbb{P}^2)$ . The first claim follows from the definition of  $\mathfrak{G}$ . The second and third claim follow by putting  $\beta := fg$  or  $\alpha = gh$  if  $\deg(f) = 1$  or  $\deg(h) = 1$ , respectively.

Assume that  $h \in \text{Aut}(\mathbb{F}_i) \setminus \text{Aut}(\mathbb{P}^2)$  and  $f \in \text{Aut}(\mathbb{F}_j) \setminus \text{Aut}(\mathbb{P}^2)$ . Since  $\text{Aut}(\mathbb{F}_0)$  is generated by  $\text{Aut}(\mathbb{P}^2) \cap \text{Aut}(\mathbb{F}_0)$  and  $\sigma_2, \sigma_3$  and  $\text{Aut}(\mathbb{F}_2)$  is generated by  $\text{Aut}(\mathbb{P}^2) \cap \text{Aut}(\mathbb{F}_2)$  and  $\sigma_1, \sigma_2$  (Lemma 2.3), we can write  $f = \beta_2 \sigma_k \alpha_2$  and  $h = \beta_1 \sigma_l \alpha_1$  for some  $\alpha_1, \beta_1 \in \text{Aut}(\mathbb{F}_i) \cap \text{Aut}(\mathbb{P}^2)$ ,  $\alpha_2, \beta_2 \in \text{Aut}(\mathbb{F}_j) \text{Aut}(\mathbb{P}^2)$  and  $k, l \in \{1, 2, 3\}$ . By replacing  $g$  with  $\alpha_2 g \beta_1$  in  $\text{Aut}(\mathbb{P}^2)$ , we can assume that  $\alpha_2 = \beta_1 = \text{Id}$ , and hence  $f = \beta_2 \sigma_k$  and  $h = \sigma_l \alpha_1$ . It follows from Remark 2.2 that

the base-points of  $f$  are exactly the base-points of  $\sigma_k$ , (\*)

the base-points of  $h^{-1}$  are exactly the base-points of  $\sigma_l$ .

(i) Suppose that  $\deg(fgh) = 1$ . Then  $f$  and  $(gh)^{-1}$  have exactly the same base-points, which are, respectively, the base-points of  $\sigma_k$  and the image of the base-points of  $\sigma_l$  by  $g$ . In particular,  $f, (gh)^{-1}$ , and hence also  $\sigma_k, \sigma_l$  have the same amount of proper base-points in  $\mathbb{P}^2$ . Since  $\sigma_1, \sigma_2, \sigma_3$  have exactly one, two and three proper base-points, it follows that  $\sigma_k = \sigma_l$ .

If  $k \in \{1, 2\}$ , then the equation  $\sigma_k = \sigma_l$ , the fact that  $f, (gh)^{-1}$  have the same base-points and (\*) imply that  $g \in \text{Aut}(\mathbb{F}_2) \cap \text{Aut}(\mathbb{P}^2)$  and so  $f, g, h \in \text{Aut}(\mathbb{F}_2)$ . The definition of  $\mathfrak{G}$  then implies  $w(f)w(g)w(h) = w(fgh)$ .

If  $k = 3$ , then the equation  $\sigma_k = \sigma_l$ , the fact that  $f, (gh)^{-1}$  have the same base-points and (\*) imply that  $g$  permutes the base-points of  $\sigma_3$ . Lemma 4.1 states that  $w(g)w(\sigma_3) = w(\sigma_3)w(\iota(g))$ . We get

$$\begin{aligned} w(f)w(g)w(h) &= w(\beta_2 \sigma_3)w(g)w(\sigma_3 \alpha_1) \\ &= w(\beta_2)w(\sigma_3)w(\sigma_3)w(\iota(g))w(\alpha_1) \\ &= w(\beta_2)w(\iota(g))w(\alpha_1) = w(\beta_2 \iota(g) \alpha_1) \\ &= w(\beta_2 \sigma_3 g \sigma_3 \alpha_1) = w(fgh). \end{aligned}$$

(ii) Suppose  $\deg(fgh) = 2$ , that is,  $f$  and  $(gh)^{-1}$  have exactly two common base-points  $s, t$ , at least one of them being proper. Assume that  $s$  is proper.

If  $t$  is infinitely near to  $s$ , (\*) implies that  $\{k, l\} \subset \{1, 2\}$ , that is,  $f, h \in \text{Aut}(\mathbb{F}_2)$ . Then (\*) and the fact that  $t$  is infinitely near  $s$  imply that  $s = [1 : 0 : 0]$  and that  $t$  lies on the strict transform of  $\{y = 0\}$ . Then  $s, t$  are base-points of both  $h^{-1}$  and  $(gh)^{-1}$  and it follows that  $g(\{s, t\}) = \{s, t\}$ , thus  $g \in \text{Aut}(\mathbb{F}_2)$ . It follows that in  $\text{Aut}(\mathbb{F}_2)$  (hence also in  $\mathfrak{G}$ )

$$w(f)w(g)w(h) = w(\beta_2 \sigma_k)w(g)w(\sigma_l \alpha_1) = w(\beta_2)w(\sigma_k g \sigma_l)w(\alpha_1).$$

Remark that any map contained in  $\text{Aut}(\mathbb{F}_2)$  is de Jonquières (Remark 3.5), from which claim (iii) follows for this subcase.

If  $s$  and  $t$  are both proper, (\*) implies that  $\{k, l\} \subset \{2, 3\}$ , that is,  $f, h \in \text{Aut}(\mathbb{F}_0)$ . Then (\*) yields that  $\{s, t\} \subset \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\}$ . There exist  $\alpha, \beta \in \text{TAut}(\mathbb{P}^2)$  such that

$$\begin{aligned} \alpha(\{[1 : 0 : 0], [0 : 1 : 0]\}) &= g^{-1}(\{s, t\}), & \beta(\{s, t\}) &= \{[1 : 0 : 0], [0 : 1 : 0]\} \\ \beta g \alpha([1 : 0 : 0]) &= [1 : 0 : 0], & \beta g \alpha([0 : 1 : 0]) &= [0 : 1 : 0]. \end{aligned}$$

If  $k = 2$ , then we may choose  $\beta = \text{Id}$ . If  $l = 2$ , then we may choose  $\alpha = \text{Id}$ . We get

$$\begin{aligned} w(f)w(g)w(h) &= w(\beta_2 \sigma_k)w(\beta^{-1})w(\beta)w(g)w(\alpha)w(\alpha^{-1})w(\sigma_l \alpha_1) \\ &\stackrel{\text{Lem 4.1}}{=} w(\beta_2 \iota(\beta^{-1}))w(\sigma_k)w(\beta g \alpha)w(\sigma_l)w(\iota(\alpha^{-1}) \alpha_1) \\ &= w(\beta_2 \iota(\beta^{-1}))w(\sigma_k \beta g \alpha \sigma_l)w(\iota(\alpha) \alpha_1). \end{aligned}$$

The claim follows with  $\alpha = \iota(\alpha^{-1}) \alpha_1$ ,  $\tilde{g} = \sigma_k(\tilde{\beta} g \tilde{\alpha}) \sigma_l$ ,  $\beta = \beta_1 \iota(\beta^{-1})$ . It remains to prove claim (iii) for this subcase: If  $f, g, h$  are de Jonquières, then  $g([1 : 0 : 0]) = [1 : 0 : 0]$  and  $[1 : 0 : 0]$  is a common base-point of  $gf$  and  $h^{-1}$ . Choosing  $\alpha, \beta$  above such that they fix  $[1 : 0 : 0]$  (that is, are de Jonquières), it follows that  $\tilde{g} = \beta g \alpha$  is de Jonquières. The maps  $f$  and  $h$  being de Jonquières implies that  $\alpha_1, \beta_2$  are de Jonquières (Remark 3.5), hence  $\iota(\alpha^{-1}) \alpha_1, \beta_1 \iota(\beta^{-1})$  and  $\tilde{g}$  are de Jonquières.  $\square$

The next two lemmata yield relations for words of length three whose image in  $\text{Bir}(\mathbb{P}^2)$  is of degree 3.

LEMMA 4.3. *Let  $f \in (\text{Aut}(\mathbb{F}_0) \cup \text{Aut}(\mathbb{F}_2)) \cap J$  be a quadratic transformation,  $\alpha_1, \dots, \alpha_4 \in \text{Aut}(\mathbb{P}^2) \cap J$  such that*

- (i)  *$f$  is a local isomorphism at the simple base-points  $q_2, q_3$  of  $(\alpha_2\sigma_3\alpha_1)^{-1}$ ;*
- (ii)  *$\text{sBp}(\alpha_4\sigma_3\alpha_3) = \{f(q_2), f(q_3)\}$ .*

Then

- (i) *the map  $(\alpha_4\sigma_3\alpha_3)f(\alpha_2\sigma_3\alpha_1)$  is quadratic de Jonquière;*
- (ii)  *$\text{sBp}((\alpha_4\sigma_3\alpha_3)f(\alpha_2\sigma_3\alpha_1)) = ((\alpha_2\sigma_3\alpha_1)^{-1})^\bullet(\text{sBp}(f))$ ;*
- (iii) *there exist  $\beta_1, \beta_3 \in \text{Aut}(\mathbb{P}^2) \cap J$  and  $\beta_2 \in \{\sigma_2, \sigma_3, \tau_{12}\sigma_2\tau_{12}\}$  such that the following equation holds in  $\mathfrak{G}$ :*

$$w(\alpha_4)w(\sigma_3)w(\alpha_3)w(f)w(\alpha_2)w(\sigma_3)w(\alpha_1) = w(\beta_3)w(\beta_2)w(\beta_1);$$

that is, the following diagram corresponds to a relation in  $\mathfrak{G}$ :

$$\begin{array}{ccc} \mathbb{P}^2 & \xrightarrow{\quad f \quad} & \mathbb{P}^2 \\ \uparrow \alpha_2\sigma_3\alpha_1 & & \downarrow \alpha_4\sigma_3\alpha_3 \\ \mathbb{P}^2 & \xrightarrow{\quad \beta_3\beta_2\beta_1 \quad} & \mathbb{P}^2 \end{array}$$

*Proof.* Define  $\tau_1 := \alpha_2\sigma_3\alpha_1$  and  $\tau_2 := \alpha_4\sigma_3\alpha_3$ , and denote by  $p_1 = [1 : 0 : 0], p_2, p_3$  the base-points of  $f$  and by  $\bar{p}_1, \bar{p}_2, \bar{p}_3$  the base-points of its inverse (ordered consistently, see Definition 3.2).

Since  $f$  is a local isomorphism at  $q_2, q_3$ , the map  $f^{-1}$  is a local isomorphism at  $f(q_2), f(q_3)$ . Hence there exist simple base-points  $p_i, \bar{p}_i$  of  $f, f^{-1}$ , respectively, either proper points of  $\mathbb{P}^2$  or infinitely near  $p_1$ , which do not lie on the lines contracted by  $(\tau_1)^{-1}$  and  $\tau_2$ . Up to order, we can assume that  $p_i = p_2$ . Therefore, the points  $\tilde{p}_2 := (\tau_1^{-1})^\bullet(p_2)$  and  $\hat{p}_2 := (\tau_2)^\bullet(\bar{p}_2)$  are proper points of  $\mathbb{P}^2$ .

Observe that the map  $\tau_2 f \tau_1$  is de Jonquière of degree 2 having base-points  $p_1, \tilde{p}_2, \tilde{p}_3 := (\tau_1^{-1})^\bullet(p_3)$  and its inverse having base-points  $p_1, \hat{p}_2, \hat{p}_3 := (\tau_2)^\bullet(\bar{p}_3)$ . Indeed, the map  $f \tau_1$  is of degree 3 with base-points  $p_1, \tilde{p}_2, \tilde{p}_3, \tilde{q}_2, \tilde{q}_3$ , where  $\tilde{q}_2, \tilde{q}_3$  are the simple base-points of  $\tau_1$ , and its inverse having base-points  $p_1, p_4, p_5, f(q_2), f(q_3)$ . Thus  $\tau_2 f \tau_1$  is de Jonquière of degree 2 with base-points  $p_1, \tilde{p}_2, \tilde{p}_3$  and its inverse having base-points  $p_1, \hat{p}_2, \hat{p}_3$  (by the formula given in Section 3).

Since  $\tau_2 f \tau_1$  has at least one simple proper base-point (namely  $\tilde{p}_2$ ), Lemmas 3.6 and 2.2 imply that there exist  $\beta_1, \beta_2 \in \text{Aut}(\mathbb{P}^2) \cap J$  and  $\beta_2 \in \{\sigma_2, \sigma_3, \tau_{12}\sigma_2\tau_{12}\}$  such that  $\tau_2 f \tau_1 = \beta_3 \beta_2 \beta_1$ .

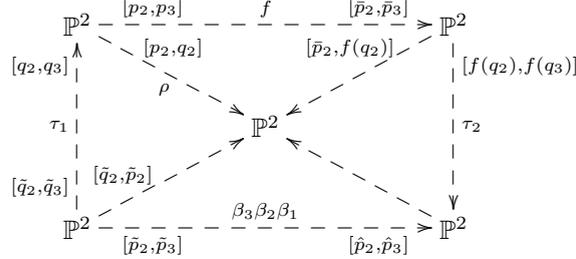
It is left to prove that  $w(\alpha_4)w(\sigma_3)w(\alpha_3)w(f)w(\alpha_2)w(\sigma_3)w(\alpha_1) = w(\beta_3)w(\beta_2)w(\beta_1)$  in  $\mathfrak{G}$ . We will use Lemma 4.2, and for this we fill the diagram

$$\begin{array}{ccc} \mathbb{P}^2 & \xrightarrow{\quad f \quad} & \mathbb{P}^2 \\ \uparrow \tau_1 & & \downarrow \tau_2 \\ \mathbb{P}^2 & \xrightarrow{\quad \beta_3\beta_2\beta_1 \quad} & \mathbb{P}^2 \end{array}$$

with triangles corresponding to relations in  $\mathfrak{G}$ .

The map  $f$  is a local isomorphism at  $q_2, q_3$ , hence the three points  $p_1, p_2, q_2$  are not collinear. Since moreover  $p_1, q_2$  are both proper points of  $\mathbb{P}^2$ , there exists a quadratic map  $\rho \in \text{Bir}(\mathbb{P}^2) \cap J$  which has base-points  $p_1, q_2, p_2$ . The maps  $\rho\tau_1$  and  $\rho f^{-1}$  are quadratic de Jonquière maps with

base-points  $p_1, \tilde{q}_2, \tilde{p}_2$  and  $p_1, \bar{p}_2, f(q_2)$ , respectively. It follows that also the map  $\rho\tau_1(\beta_3\beta_2\beta_1)^{-1}$  is quadratic. The situation is summarized in the following diagram, where all the arrows are quadratic maps and the points in the brackets are the simple base-points of the corresponding quadratic map contained in  $J$ :



Writing  $\rho = \gamma_3\gamma_2\gamma_1$  for some  $\gamma_1, \gamma_2, \gamma_3 \in (\text{Aut}(\mathbb{P}^2) \cup \text{Aut}(\mathbb{F}_0) \cup \text{Aut}(\mathbb{F}_2)) \cap J$ , only  $\gamma_2$  quadratic (possible by Lemma 3.6), Lemma 4.2 implies that each triangle in the above diagram corresponds to a relation in  $\mathfrak{G}$ , making the whole diagram correspond to a relation in  $\mathfrak{G}$ .  $\square$

LEMMA 4.4. *Let  $f, h \in \text{Aut}(\mathbb{F}_0) \cup \text{Aut}(\mathbb{F}_2)$ ,  $g \in \text{Aut}(\mathbb{P}^2)$ ,  $f, g, h \in J$ , and let  $\Delta$  be a de Jonquières linear system. Assume that*

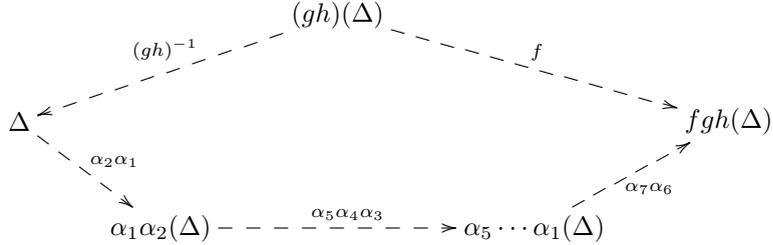
$$\deg(fgh) = 3, \quad \deg(fgh(\Delta)) < \deg(gh(\Delta)), \quad \deg(\Delta) \leq \deg(gh(\Delta))$$

*and that  $(gh)(\Delta)$  has a proper base-point different from  $[1 : 0 : 0]$ . Then there exist  $\alpha_1, \dots, \alpha_7 \in (\text{Aut}(\mathbb{P}^2) \cup \text{Aut}(\mathbb{F}_0) \cup \text{Aut}(\mathbb{F}_2)) \cap J$ ,  $\alpha_1, \alpha_3, \alpha_5, \alpha_7 \in \text{Aut}(\mathbb{P}^2)$ , such that*

(i) *the following equation holds in  $\mathfrak{G}$ :*

$$w(f)w(g)w(h) = w(\alpha_7) \cdots w(\alpha_1)$$

*that is, the following diagram corresponds to a relation in  $\mathfrak{G}$ :*



(ii) *for  $i = 2, \dots, 7$*

$$\deg(\alpha_i \cdots \alpha_1(\Delta)) < \deg((gh)(\Delta)).$$

*Proof.* The equality  $\deg(fgh) = 3$  implies that  $f$  and  $(gh)^{-1}$  have exactly one common base-point, namely  $p_1 = [1 : 0 : 0]$ . Denote by  $\text{sBp}((gh)^{-1}) = \{p_2, p_3\}$  and  $\text{sBp}(f) = \{p_4, p_5\}$  the simple base-points of  $(gh)^{-1}$  and  $f$ , respectively, and write  $d = \deg(gh(\Delta))$ .

By assumption,  $gh(\Delta)$  is a de Jonquières linear system that has a proper base-point  $s$  different from  $p_1$ . For any point  $r$ , let  $m(r)$  be the multiplicity of  $gh(\Delta)$  in  $r$ , respectively. Then  $m(p_1) = d - 1$  and  $m(s) = 1$  (Remark 3.5), and Remark 3.7 implies that because  $\deg(\Delta) \leq d$  and  $\deg(fgh(\Delta)) < d$ , we have (up to ordering of  $p_2, p_3$ )

$$\begin{aligned} m(p_2) &= 1, & m(p_3) &\leq 1, \\ \deg(fgh(\Delta)) &= d - 1, & m(p_4) &= m(p_5) = 1. \end{aligned}$$

We will now construct  $\alpha_1, \dots, \alpha_7$ .

Assume that  $s \in \{p_2, p_3, p_4, p_5\}$ . If  $s \in \{p_2, p_3\}$ , then we choose  $r \in \{p_4, p_5\}$ . If  $s \in \{p_4, p_5\}$ , then we choose  $r \in \{p_i : i = 2, 3 \text{ and } m(p_i) = 1\}$ . We choose  $r$  to be infinitely near  $p_1$  or a proper point (this is always possible). The points  $p_1, s, r$  are not aligned, because  $a_1 + m(s) + m(r) > d$ , thus there exist  $\rho \in \text{Bir}(\mathbb{P}^2)$  quadratic de Jonquières with base-points  $p_1, r, s$ . The following commutative diagram, where the points in the brackets are the base-points of the corresponding map, summarizes the situation:

$$\begin{array}{ccccc}
 & & [p_1, p_2, p_3] & gh(\Delta) & [p_1, p_4, p_5] \\
 & & \swarrow & \downarrow & \searrow \\
 & & (gh)^{-1} & [p_1, s, r] & \rho & f \\
 & & \swarrow & \downarrow & \searrow \\
 \Delta & \xleftarrow{\quad} & \rho gh(\Delta) & \xrightarrow{\quad} & fgh(\Delta)
 \end{array}$$

Using Remark 3.7, we obtain

$$\begin{aligned}
 \deg(\rho gh) &= \deg(f\rho^{-1}) = 2, \\
 \deg(\rho gh(\Delta)) &= d - 1 < \deg(gh(\Delta)).
 \end{aligned}$$

We write  $\rho = \gamma\tilde{\rho}\delta$ ,  $\rho gh = \alpha_3\alpha_2\alpha_1$ ,  $f\rho^{-1} = \alpha_6\alpha_5\alpha_4$ , where  $\delta, \gamma, \alpha_1, \dots, \alpha_6 \in (\text{Aut}(\mathbb{P}^2) \cup \text{Aut}(\mathbb{F}_0) \cup \text{Aut}(\mathbb{F}_2)) \cap J$ , only  $\tilde{\rho}, \alpha_2, \alpha_5$  quadratic (Lemma 3.6). By Lemma 4.2, the above diagram is generated by relations in  $\mathfrak{G}$ . Hence  $w(f)w(g)w(h) = w(\alpha_6) \cdots w(\alpha_1)$  in  $\mathfrak{G}$ .

Assume  $s \notin \{p_2, p_3, p_4, p_5\}$ ; we choose  $r_1 \in \{p_i : i = 2, 3 \text{ and } m(p_i) = 1\}$ ,  $r_2 \in \{p_4, p_5\}$  such that  $r_1$  (respectively,  $r_2$ ) is either a proper point or infinitely near  $p_1$  (this is always possible). For  $i = 1, 2$ , the points  $p_1, s, r_i$  are not collinear, because  $a_1 + m(s) + m(r_i) > d$ . Thus there exist  $\rho_1, \rho_2 \in \text{Bir}(\mathbb{P}^2)$  quadratic de Jonquières with base-points  $p_1, s, r_1$  and  $p_1, s, r_2$ , respectively. The following commutative diagram, where the brackets are the base-points of the corresponding map, summarizes the situation:

$$\begin{array}{ccccccc}
 & & & gh(\Delta) & & & \\
 & & [p_1, p_2, p_3] & \swarrow & \searrow & [p_1, p_4, p_5] & \\
 & & \swarrow & \downarrow & \downarrow & \searrow & \\
 & & (gh)^{-1} & [p_1, s, r_1] & \rho_1 & [p_1, s, r_2] & \rho_2 & f \\
 & & \swarrow & \downarrow & \downarrow & \searrow & \searrow & \\
 \Delta & \xleftarrow{\quad} & \rho_1 gh(\Delta) & \xrightarrow{\quad} & \rho_2 gh(\Delta) & \xrightarrow{\quad} & fgh(\Delta)
 \end{array}$$

Using Remark 3.7, we obtain

$$\begin{aligned}
 \deg(\rho_1 gh) &= \deg(\rho_2 \rho_1^{-1}) = \deg(f\rho_2^{-1}) = 2, \\
 \deg(\rho_1 gh(\Delta)) &= d - 1 < \deg(gh(\Delta)), \\
 \deg(\rho_2 gh(\Delta)) &= d - 1 < \deg(gh(\Delta)).
 \end{aligned}$$

We write  $\rho_1 = \gamma_1\tilde{\rho}_1\beta_1$ ,  $\rho_2 = \gamma_2\tilde{\rho}_2\beta_2$ ,  $\rho_1 gh = \alpha_3\alpha_2\alpha_1$ ,  $\rho_2 \rho_1^{-1} = \alpha_6\alpha_5\alpha_4$ ,  $f\rho_2^{-1} = \alpha_9\alpha_8\alpha_7$  for  $\alpha_1, \dots, \alpha_9, \beta_1, \beta_2, \gamma_1, \gamma_2, \tilde{\rho}_1, \tilde{\rho}_2 \in (\text{Aut}(\mathbb{P}^2) \cup \text{Aut}(\mathbb{F}_0) \cup \text{Aut}(\mathbb{F}_2)) \cap J$ , only  $\alpha_2, \alpha_5, \alpha_8, \tilde{\rho}_1, \tilde{\rho}_2$  quadratic (Lemma 3.6). Lemma 4.2 implies that all triangles of the above diagram are generated by relations in  $\mathfrak{G}$ , and thus  $w(f)w(g)w(h) = w(\alpha_9) \cdots w(\alpha_1)$  in  $\mathfrak{G}$ . We obtain the maps  $\alpha_i$  in the claim by merging neighbour automorphisms of  $\mathbb{P}^2$  in the product  $\alpha_9 \cdots \alpha_1$ .  $\square$

## 5. The Cremona group is isomorphic to $\mathfrak{G}$

In this section, we prove Theorem B (Theorem 5.5). The main tool will be Lemma 5.1, which yields the existence of an injective map  $w_J: J \rightarrow \mathfrak{G}$  such that  $\pi \circ w_J = \text{Id}$  (Corollary 5.2) and enables us to use the result (Theorem 5.3) of [4], that  $\text{Bir}(\mathbb{P}^2)$  is isomorphic to the

amalgamated product of  $\text{Aut}(\mathbb{P}^2)$  and  $J$  along their intersection modulo one relation, for the proof of Theorem B (Theorem 5.5).

LEMMA 5.1. *Let  $f_1, \dots, f_n \in (\text{Aut}(\mathbb{P}^2) \cup \text{Aut}(\mathbb{F}_0) \cup \text{Aut}(\mathbb{F}_2)) \cap J$  such that  $f_n \cdots f_1 = \text{Id}$ . Then  $w(f_n) \cdots w(f_1) = \text{Id}$  in  $\mathfrak{G}$ .*

*Proof.* We can write  $w(f_n) \cdots w(f_1) = w(\alpha_{m+1})w(g_m)w(\alpha_m) \cdots w(\alpha_2)w(g_1)w(\alpha_1)$ , where  $\alpha_i \in \text{Aut}(\mathbb{P}^2) \cap J$  and  $g_i \in (\text{Aut}(\mathbb{F}_0) \cup \text{Aut}(\mathbb{F}_2)) \cap J \setminus \text{Aut}(\mathbb{P}^2)$  as follows: We put  $g_j := f_j$  if  $f_j$  is quadratic,  $\alpha_j := f_j$  if  $f_j$  is linear. Then we proceed by putting  $\alpha_j := \alpha_{i+1}\alpha_i$  ( $w(\alpha_j) = w(\alpha_{i+1}\alpha_i)$ ) by Lemma 4.2). Proceeding like this, we will reach a word where no two consecutive letters both have linear image in  $\text{Bir}(\mathbb{P}^2)$ . We then insert  $\alpha_j = \text{Id}$  between any two consecutive letters whose both image in  $\text{Bir}(\mathbb{P}^2)$  is quadratic.

We denote by  $\Delta_0$  the linear system of lines in  $\mathbb{P}^2$  and define, for  $i = 1, \dots, m$ ,

$$\Delta_i := (\alpha_i g_{i-1} \cdots g_1 \alpha_1)(\Delta_0)$$

which is the linear system of the map  $(\alpha_i g_{i-1} \cdots g_1 \alpha_1)^{-1}$ . We define  $d_i := \deg(\Delta_i)$ , which is also the degree of the map  $(\alpha_i g_{i-1} \cdots g_1 \alpha_1)^{-1}$ . Furthermore, we define

$$D := \max\{d_i \mid i = 1, \dots, m\}, \quad N := \max\{i \mid d_i = D\}.$$

If  $D = 1$ , then it follows that  $m = 1$  and  $\alpha_1 = \text{Id}$ . We can therefore assume that  $D > 1$  and prove the result by induction over the lexicographically ordered pair  $(D, N)$ .

The induction step consists of finding  $\tilde{\alpha}_{k+1}, \dots, \tilde{\alpha}_1 \in \text{Aut}(\mathbb{P}^2) \cap J$  and  $\tilde{g}_1, \dots, \tilde{g}_k \in (\text{Aut}(\mathbb{F}_0) \cup \text{Aut}(\mathbb{F}_2)) \cap J \setminus \text{Aut}(\mathbb{P}^2)$  such that

$$w(g_{N+1})w(\alpha_{N+1})w(g_N) = w(\tilde{\alpha}_{k+1})w(\tilde{g}_k) \cdots w(\tilde{g}_1)w(\tilde{\alpha}_1)$$

and such that the pair  $(\tilde{D}, \tilde{N})$  associated to the product

$$\alpha_{m+1}g_m \cdots g_{N+2}(\alpha_{N+2}\tilde{\alpha}_{k+1})\tilde{g}_k \cdots \tilde{g}_1(\tilde{\alpha}_1\alpha_N)g_{N-1} \cdots g_1\alpha_1$$

is strictly smaller than  $(D, N)$ .

We look at three cases, depending on the degree of  $g_{N+1}\alpha_{N+1}g_N$ , and if the degree is 3, we look at two subcases, the ‘good case’ and the ‘bad case’.

If  $\deg(g_{N+1}\alpha_{N+1}g_N) = 1$ , then define  $\text{Aut}(\mathbb{P}^2) \ni \tilde{\alpha} := g_{N+1}\alpha_{N+1}g_N$ . It follows from Lemma 4.2 (1) that  $w(g_{N+1})w(\alpha_{N+1})w(g_N) = w(\tilde{\alpha})$  in  $\mathfrak{G}$ . We replace  $g_{N+1}\alpha_{N+1}g_N$  by  $\tilde{\alpha}$ , which decreases  $(D, N)$ .

If  $\deg(g_{N+1}\alpha_{N+1}g_N) = 2$ , then it follows from Lemma 4.2 (2),(3) that there exist  $\tilde{\alpha}, \tilde{\beta} \in \text{Aut}(\mathbb{P}^2) \cap J$  and  $\tilde{g} \in (\text{Aut}(\mathbb{F}_0) \cup \text{Aut}(\mathbb{F}_2)) \cap J \setminus \text{Aut}(\mathbb{P}^2)$  such that  $w(g_{N+1})w(\alpha_{N+1})w(g_N) = w(\tilde{\beta})w(\tilde{g})w(\tilde{\alpha})$ . We replace  $g_{N+1}\alpha_{N+1}g_N$  by  $\tilde{\beta}\tilde{g}\tilde{\alpha}$ , which decreases  $(D, N)$ .

Finally, suppose that  $\deg(g_{N+1}\alpha_{N+1}g_N) = 3$ . By definition of  $N$ , we have

$$d_{N-1} \leq D, \quad d_N = D, \quad d_{N+1} < D.$$

‘Good case’: If  $\Delta_N$  has a proper simple base-point, then it follows from Lemma 4.4 (with  $\Delta = \Delta_{N-1}$ ) that there exist  $\tilde{\alpha}_1, \dots, \tilde{\alpha}_4 \in \text{Aut}(\mathbb{P}^2) \cap J$ ,  $\tilde{g}_1, \tilde{g}_2, \tilde{g}_3 \in (\text{Aut}(\mathbb{F}_0) \cup \text{Aut}(\mathbb{F}_2)) \cap J$  such that

$$w(g_{N+1})w(\alpha_{N+1})w(g_N) = w(\tilde{\alpha}_4)w(\tilde{g}_3) \cdots w(\tilde{\alpha}_2)w(\tilde{g}_1)w(\tilde{\alpha}_1) \quad \text{in } \mathfrak{G}$$

and

$$\deg((\tilde{\alpha}_{i+1}\tilde{g}_i \cdots g_1\tilde{\alpha}_1)(\Delta_{N-1})) < \deg(\Delta_N) = D$$

for  $i = 1, \dots, 4$ . Replacing  $g_{N+1}\alpha_{N+1}g_N$  by  $\tilde{\alpha}_4\tilde{g}_3 \cdots \tilde{g}_1\tilde{\alpha}_1$  decreases  $(D, N)$ .

‘Bad case’: Assume that  $\Delta_N$  has no simple proper base-points. Without changing the pair  $(D, N)$ , we will replace the word  $w(\alpha_{m+1})w(g_m) \cdots w(g_1)w(\alpha_1)$  in  $\mathfrak{G}$  by an equivalent word  $w(\hat{\alpha}_{m+1})w(\hat{g}_m) \cdots w(\hat{g})w(\hat{\alpha}_1)$  satisfying the ‘good case’.

Choose two general points  $p_0, q_0$  in  $\mathbb{P}^2$  and write  $p_1 = (\alpha_2 g_1 \alpha_1)(p_0)$ ,  $q_1 = (\alpha_2 g_1 \alpha_1)(q_0)$  and  $p_i = (\alpha_{i+1} g_i)(p_{i-1})$ ,  $q_i = (\alpha_{i+1} g_i)(q_{i-1})$  for  $i = 2, \dots, m$ . Note that  $p_m = p_0$  and  $q_m = q_0$  because  $\alpha_{m+1} g_m \cdots g_1 \alpha_1 = \text{Id}$ .

For  $i = 0, \dots, m$ , we denote by  $\beta_i \in \text{Aut}(\mathbb{P}^2)$  an element sending  $[1 : 0 : 0]$ ,  $[0 : 1 : 0]$ ,  $[0 : 0 : 1]$ , respectively, onto  $[1 : 0 : 0]$ ,  $p_i, q_i$  (this is possible, because we took  $p_0, q_0$  general), and write  $\tau_i := \beta_i \sigma_3 (\beta_i)^{-1}$ , which is a quadratic de Jonquières involution having base-points  $[1 : 0 : 0]$ ,  $p_i, q_i$ . We choose  $\beta_m = \beta_0$  and then have  $\tau_m = \tau_0$ .

By Lemma 4.3, the maps  $\tau_1(\alpha_2 g_1 \alpha_1) \tau_0^{-1}$ ,  $\tau_i(g_i \alpha_i) \tau_{i-1}^{-1}$  are quadratic de Jonquières and there exist  $\gamma_i, \delta_i \in \text{Aut}(\mathbb{P}^2) \cap J$ ,  $\hat{g}_i \in \{\sigma_2, \sigma_3, \tau_{12} \sigma_2 \tau_{12}\}$  such that

$$\begin{aligned} w(\beta_1)w(\sigma_3)w(\beta_1^{-1})w(\alpha_2)w(g_1)w(\alpha_1)w(\beta_0)w(\sigma_3)w(\beta_0^{-1}) &= w(\delta_1)w(\hat{g}_1)w(\gamma_1), \\ w(\beta_i)w(\sigma_3)w(\beta_i^{-1})w(\alpha_{i+1})w(g_i)w(\beta_{i-1})w(\sigma_3)w(\beta_{i-1}^{-1}) &= w(\delta_i)w(\hat{g}_i)w(\gamma_i) \end{aligned}$$

for  $i = 1, \dots, m$ . We get the following diagram

$$\begin{array}{ccccc} \begin{array}{ccc} \xrightarrow{\alpha_2 g_1 \alpha_1} & \xrightarrow{\alpha_3 g_2} & \dots \\ \tau_0 \downarrow & \downarrow \tau_1 & \downarrow \tau_2 \\ \xrightarrow{\delta_1 \hat{g}_1 \gamma_1} & \xrightarrow{\delta_2 \hat{g}_2 \gamma_2} & \dots \end{array} & & \begin{array}{ccc} \xrightarrow{\alpha_{i+1} g_i} & & \dots \\ \tau_{i-1} \downarrow & \downarrow \tau_i & \\ \xrightarrow{\delta_i \hat{g}_i \gamma_i} & & \dots \end{array} & & \begin{array}{ccc} \xrightarrow{\alpha_{m+1} g_m} & & \\ \tau_{m-1} \downarrow & \downarrow \tau_m & \\ \xrightarrow{\delta_m \hat{g}_m \gamma_m} & & \end{array} \end{array}$$

where each square in the diagram corresponds to a relation in  $\mathfrak{G}$ , making the whole diagram correspond to a relation in  $\mathfrak{G}$ . Therefore, writing  $\tilde{\alpha}_i := \delta_i \gamma_{i-1}$  for  $i = 2, \dots, m$ ,  $\tilde{\alpha}_{m+1} := \delta_m$ ,  $\tilde{\alpha}_1 := \gamma_1$ , the equality

$$w(\alpha_{m+1})w(g_m) \cdots w(g_1)w(\alpha_1) = w(\hat{\alpha}_{m+1})w(\hat{g}_m)w(\hat{\alpha}_m) \cdots w(\hat{\alpha}_2)w(\hat{g}_1)w(\hat{\alpha}_1)$$

holds in  $\mathfrak{G}$ . We replace  $\alpha_{m+1} g_m \cdots g_1 \alpha_1$  by  $\hat{\alpha}_{m+1} \hat{g}_m \hat{\alpha}_m \cdots \hat{\alpha}_2 \hat{g}_1 \hat{\alpha}_1$ .

For  $i = 1, \dots, m$ , call  $\hat{\Delta}_i := (\hat{\alpha}_i \hat{g}_{i-1} \cdots \hat{g}_1 \hat{\alpha}_1)(\Delta_0)$ , which is the linear system of the map  $(\hat{\alpha}_i \hat{g}_{i-1} \cdots \hat{g}_1 \hat{\alpha}_1)^{-1}$ , and denote by  $\hat{d}_i$  its degree. Using Remark 3.7, we get  $\deg(\hat{\alpha}_i \hat{g}_{i-1} \cdots \hat{g}_1 \hat{\alpha}_1) = \deg(\alpha_i g_{i-1} \cdots g_1 \alpha_1)$  for each  $i$ , thus  $\hat{d}_i = d_i$  for  $i = 1, \dots, m$ . Therefore, the replacement does not change the pair  $(D, N)$ , that is,  $(\hat{D}, \hat{N}) = (D, N)$ .

It remains to show that  $\hat{\alpha}_{n+1} \hat{g}_n \cdots \hat{g}_1 \hat{\alpha}_1$  satisfies the ‘good case’, that is, that  $\hat{\Delta}_N$  has a simple proper base-point.

Since  $d_{N+1} < D$ , it follows from Remark 3.7 that  $d_{N+1} = D - 1$  and that all the base-points of  $g_{N+1}$  are base-points of  $\Delta_N$ . Since the base-points of  $\tau_N$  are general, it follows from Remark 3.7 that each point in  $(\tau_N)^\bullet(\text{sBp}(g_N))$  is a base-point of  $\hat{\Delta}_N$ . Lemma 4.3 states that  $(\tau_N)^\bullet(\text{sBp}(g_N)) = \text{sBp}(\hat{g}_N)$ , and  $\hat{g}_N \in \{\sigma_2, \sigma_3, \tau_{12} \sigma_2 \tau_{12}\}$  has a simple proper base-point. Hence  $\hat{\Delta}_N$  has a simple proper base-point.  $\square$

**COROLLARY 5.2.** *Let  $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m \in \text{Aut}(\mathbb{P}^2) \cup \text{Aut}(\mathbb{F}_0) \cup \text{Aut}(\mathbb{F}_2)$  de Jonquières such that  $\alpha_n \cdots \alpha_1 = \beta_m \cdots \beta_1$ . Then*

$$w(\alpha_n) \cdots w(\alpha_1) = w(\beta_m) \cdots w(\beta_1).$$

*In particular, there exists a homomorphism  $w_J: J \rightarrow \mathfrak{G}$  which sends  $\alpha_n \cdots \alpha_1$  onto  $w(\alpha_n) \cdots w(\alpha_1)$  and  $\pi \circ w_J = \text{Id}$ , that is,  $w_J$  is injective.*

*Proof.* The claim follows from applying Lemma 5.1 to  $\beta_1^{-1} \cdots \beta_m^{-1} \alpha_n \cdots \alpha_1$ .  $\square$

**PROPOSITION 5.3 [4].** *The group  $\text{Bir}(\mathbb{P}^2)$  is isomorphic to*

$$(\text{Aut}(\mathbb{P}^2) *_{\text{Aut}(\mathbb{P}^2) \cap J} J) / \langle \tau_{12} \sigma_3 \tau_{12} \sigma_3 \rangle,$$

*the amalgamated product of  $\text{Aut}(\mathbb{P}^2)$  and  $J$  along their intersection and divided by the relation  $\tau_{12} \sigma_3 = \sigma_3 \tau_{12}$ , where  $\tau_{12}([x : y : z]) = [y : x : z]$ .*

REMARK 5.4. In  $\text{Bir}(\mathbb{P}^2)$ , the three relations

- (i)  $\tau_{12}\sigma_3\tau_{12}\sigma_3 = \text{Id}$ ;
- (ii)  $\tau_{13}\sigma_3\tau_{13}\sigma_3 = \text{Id}$ ;
- (iii)  $\tau_{23}\sigma_3\tau_{23}\sigma_3 = \text{Id}$

hold. Choosing two of them, the remaining relation of the three is generated by the chosen two. Relation (iii) is a relation holding in  $J$ . Thus it suffices to impose relation (i) or (ii) in Theorem 5.3.

Since  $\tau_{12}, \sigma_3 \in \text{Aut}(\mathbb{F}_0)$ , relation (i) holds in  $\text{Aut}(\mathbb{F}_0)$ , so in particular it holds in the generalized amalgamated product of  $\text{Aut}(\mathbb{P}^2)$ ,  $\text{Aut}(\mathbb{F}_0)$ ,  $\text{Aut}(\mathbb{F}_2)$  along all their pairwise intersections. It is *a priori* not clear whether or not one of the relations (ii), (iii) holds in the generalized amalgamated product of  $\text{Aut}(\mathbb{P}^2)$ ,  $\text{Aut}(\mathbb{F}_0)$ ,  $\text{Aut}(\mathbb{F}_2)$  along all their pairwise intersections because it is not clear whether or not  $J$  embeds into it or not. Therefore, we need to impose one of the relations (ii) or (iii).

THEOREM 5.5 (Theorem B). *The group  $\text{Bir}(\mathbb{P}^2)$  is isomorphic to  $\mathfrak{G}$ , the generalized amalgamated product of  $\text{Aut}(\mathbb{P}^2)$ ,  $\text{Aut}(\mathbb{F}_0)$ ,  $\text{Aut}(\mathbb{F}_2)$  along all the pairwise intersections modulo the relation  $\tau_{13}\sigma_3\tau_{13}\sigma_3$ , where  $\tau_{13}([x : y : z]) = [z : y : x]$ .*

*Proof.* By Corollary 5.2, there exists  $w_J : J \rightarrow \mathfrak{G}$  such that  $\pi \circ w_J = \text{Id}$ , and  $w$  and  $w_J$  coincide on  $\text{Aut}(\mathbb{P}^2) \cap J$ . Thus the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{G} & \xleftarrow{w_J} & J \\ w \uparrow & & \downarrow \iota_2 \\ \text{Aut}(\mathbb{P}^2) & \xleftarrow{\iota_1} & \text{Aut}(\mathbb{P}^2) \cap J \end{array}$$

where  $\iota, \iota_1$  are the canonical inclusion maps. The universal property of the amalgamated product implies the existence of a unique homomorphism  $\varphi : \text{Aut}(\mathbb{P}^2) *_{\text{Aut}(\mathbb{P}^2) \cap J} J \rightarrow \mathfrak{G}$  such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{G} & & \\ \varphi \nearrow & \xleftarrow{w_J} & J \\ \exists! \downarrow & & \downarrow \iota \\ \text{Aut}(\mathbb{P}^2) *_{\text{Aut}(\mathbb{P}^2) \cap J} J & \xleftarrow{\iota} & \text{Aut}(\mathbb{P}^2) \cap J \\ w \nearrow & & \downarrow \iota \\ \text{Aut}(\mathbb{P}^2) & \xleftarrow{\iota} & \text{Aut}(\mathbb{P}^2) \cap J \end{array}$$

By Proposition 5.3,  $\text{Bir}(\mathbb{P}^2)$  is isomorphic to  $\text{Aut}(\mathbb{P}^2) *_{\text{Aut}(\mathbb{P}^2) \cap J} J$  modulo the relation  $\sigma_3\tau_{12} = \tau_{12}\sigma_3$ , where  $\tau_{12}([x : y : z]) = [y : x : z]$ . Since  $\tau_{12}, \sigma_3 \in \text{Aut}(\mathbb{F}_0)$ , the relation  $\sigma_3\tau_{12} = \tau_{12}\sigma_3$  also holds in  $\text{Aut}(\mathbb{F}_0)$ , and hence in  $\mathfrak{G}$ . Thus, the homomorphism  $\varphi$  induces a homomorphism  $\bar{\varphi} : (\text{Aut}(\mathbb{P}^2) *_{\text{Aut}(\mathbb{P}^2) \cap J} J) / \langle \sigma_3\tau_{12}\sigma_3\tau_{12} \rangle \rightarrow \mathfrak{G}$ . By construction,  $\bar{\varphi}$  and the canonical homomorphism  $\pi : \mathfrak{G} \rightarrow \text{Bir}(\mathbb{P}^2)$  are inverse to each other.  $\square$

## 6. The Cremona group is compactly presented

In this section, we restrict to case  $k = \mathbb{C}$  and show that  $\text{Bir}(\mathbb{P}^2)$  is compactly presented using Theorem 5.5 (Theorem B).

Being compactly presented is a notion reserved for Hausdorff topological groups and we consider  $\text{Bir}(\mathbb{P}^2)$  endowed with the Euclidean topology as constructed in [5, Section 5], which

makes  $\text{Bir}(\mathbb{P}^2)$  a Hausdorff topological group [5, Theorem 3], which is not locally compact [5, Lemma 5.15].

DEFINITION 6.1. Let  $G$  be a group.

(i) A presentation  $\langle S \mid R \rangle$  of  $G$  is a triple made up of a set  $S$ , an epimorphism  $\pi : F_S \twoheadrightarrow G$  of the free group on  $S$  onto  $G$ , a subset  $R$  of  $F_S$  generating  $\ker(\pi)$  as a normal subgroup. The relations of the presentation are the elements of  $\ker(\pi)$  and the elements of  $R$  the *relators* (or *generating relations*) of the presentation.

(ii) A *bounded presentation* of  $G$  is a presentation  $\langle S \mid R \rangle$  of  $G$  with  $R$  a set of relators of bounded length.

(iii) Let  $G$  be a Hausdorff topological group. A *compact presentation* of  $G$  is a presentation  $\langle S \mid R \rangle$  of  $G$  with  $S$  a compact subset of  $G$  and  $R$  a set of relators of bounded length. We say that  $G$  is *compactly presented by  $S$*  if  $G$  is given by a compact presentation  $\langle S \mid R \rangle$ . We also say that  $G$  is *compactly presented* if  $G$  is compactly presented by some subset.

LEMMA 6.2. (i) Let  $G$  be a group and  $S_1, S_2 \subset G$  be generating subsets. If  $S_1^m \subset S_2^n \subset S_1^{m'}$  for some  $m, n, m' \in \mathbb{N}$ , then  $G$  is boundedly presented by  $S_1$  if and only if it is boundedly presented by  $S_2$ .

(ii) Any connected topological group is generated by any neighbourhood of 1.

(iii) If  $G$  is a locally compact Hausdorff topological group having only finitely many connected components, then it is compactly presented.

(iv) If  $G$  is a locally compact Hausdorff topological group that is compactly presented, then it is compactly presented by all its compact generating subsets.

(v) Let  $G$  be a locally compact topological group with finitely many connected components  $G_0, \dots, G_n$ , where  $1 \in G_0$ . For each  $i$  choose some  $g_i \in G$  such that  $G_i = g_i G_0$ . Then  $G$  is generated by any compact neighbourhood of 1 and  $g_1, \dots, g_n$ . In particular, it is compactly presented by any compact neighbourhood of 1 and  $g_1, \dots, g_n$ .

*Proof.* (i) is proved in the forthcoming paper ‘Metric geometry of locally compact groups’, by de Cornulier and de la Harpe (<http://www.normalesup.org/cornulier/MetricLC.pdf>), Lemma 7.A.9) and [9, Lemma 2.6] and (iii) in [1, Satz 3.2] (see also the forthcoming paper ‘Metric geometry of locally compact groups’, by de Cornulier and de la Harpe (<http://www.normalesup.org/cornulier/MetricLC.pdf>), Subsection 8.A).

(ii) Let  $U \subset G$  be an open neighbourhood of 1. Then the subgroup  $H$  of  $G$  generated by  $U$  is open because  $H = \bigcup_{h \in H} hU$ . It is also closed because  $G \setminus H = \bigcup_{g \in G \setminus H} gH$ , which is an open set.

(iii) If  $G$  is compactly generated by a compact set  $S$  and  $K \subset G$  is a compact set, then  $K \subset S^n$  for some large  $n$ . This follows from the fact that any locally compact topological group is a Baire space and that  $S$  is compact. The claim now follows from (i).

(iv) Let  $K \subset G_0$  be a compact neighbourhood of 1. By (ii),  $K$  generates  $G_0$ , and thus the compact set  $K \cup \{g_1, \dots, g_n\}$  generates  $G$ . By (iii) and (iv), the locally compact group  $G$  is compactly presented by  $K \cup \{g_1, \dots, g_n\}$ .  $\square$

REMARK 6.3. Any irreducible algebraic variety over  $\mathbb{C}$  is connected with respect to the Euclidean topology [12, Chapter XII, Proposition 2.4]. Any linear algebraic subgroup of  $\text{Bir}(\mathbb{P}^2)$  has finitely many irreducible components in the Zariski topology, which are exactly the cosets of the component containing 1. Thus they are the connected components in the Zariski topology, and hence also the connected components in the Euclidean topology.

Furthermore, any linear algebraic subgroup of  $\text{Bir}(\mathbb{P}^2)$  is a closed subset of  $\text{Bir}(\mathbb{P}^2)_{\leq d}$  for some  $d \in \mathbb{N}$  (see [5, Lemma 2.19]), which is a locally compact set [5, Lemma 5.4]. Hence any linear algebraic subgroup of  $\text{Bir}(\mathbb{P}^2)$  is locally compact and therefore satisfies the conditions of Lemma 6.2(v).

REMARK 6.4. Any algebraic subgroup of  $\text{Bir}(\mathbb{P}^2)$  is a linear algebraic group [3, Théorème 2]. The Euclidean topology on these groups is exactly the topology inherited from the Euclidean topology on  $\text{Bir}(\mathbb{P}^2)$  (see [5, Proposition 5.11]).

The groups  $\text{Aut}(\mathbb{P}^2) = \text{PGL}_3(\mathbb{C})$ ,  $\text{Aut}(\mathbb{F}_0)$  and  $\text{Aut}(\mathbb{F}_2)$  are linear algebraic subgroups of  $\text{Bir}(\mathbb{P}^2)$  (Lemma 2.4), and thus locally compact by Remark 6.3.

COROLLARY 6.5. (i) *The group  $\text{Aut}(\mathbb{P}^2)$  is compactly presented by any compact neighbourhood of 1.*

(ii) *The group  $\text{Aut}(\mathbb{F}_0)$  is compactly presented by the union of the linear map  $\tau_{12}: [x : y : z] \rightarrow [y : x : z]$  and any compact neighbourhood of 1.*

(iii) *The group  $\text{Aut}(\mathbb{F}_2)$  is compactly presented by any compact neighbourhood of 1.*

*Proof.* The groups  $\text{Aut}(\mathbb{P}^2)$ ,  $\text{Aut}(\mathbb{F}_0)$ ,  $\text{Aut}(\mathbb{F}_2)$  are linear algebraic groups and locally compact by Remark 6.4.

The group  $\text{Aut}(\mathbb{P}^2) = \text{PGL}_3(k)$  is irreducible, hence connected (Remark 6.3), and the group  $\text{Aut}(\mathbb{F}_2)$  is connected by Lemma 2.4(iii). By Lemma 2.4(ii), the group  $\text{Aut}(\mathbb{F}_0)$  has two connected components, namely  $\text{Aut}(\mathbb{F}_0)^0$  containing the identity element and  $\tau_{12}\text{Aut}(\mathbb{F}_0)^0$ . The claim now follows from Remark 6.3 and Proposition 6.2(v).  $\square$

Using Corollary 6.5 and the fact that  $\text{Bir}(\mathbb{P}^2)$  is isomorphic to the generalized amalgamated product of  $\text{Aut}(\mathbb{P}^2)$ ,  $\text{Aut}(\mathbb{F}_0)$ ,  $\text{Aut}(\mathbb{F}_2)$  along their pairwise intersection divided by one relation (Theorem 5.5), we prove that  $\text{Bir}(\mathbb{P}^2)$  is compactly presentable.

LEMMA 6.6. *Let  $G$  be a group,  $n \geq 2$  be an integer and  $G_1, \dots, G_n \subset G$  be subgroups of  $G$  such that the following hold.*

- (i) *The group  $G$  admits the presentation  $G = \langle \bigcup_{i=1}^n G_i \mid R_G \rangle$ , where  $R_G$  is the set of all relators of the form  $ab = c$ , where  $a, b, c \in G_i$  for some  $i \in \{1, \dots, n\}$ .*
- (ii) *For  $i = 1, \dots, n$ , there exists a presentation  $\langle K_i \mid R_i \rangle$  of  $G_i$  such that, for any subset  $I \subset \{1, \dots, n\}$ , the set  $\bigcap_{i \in I} K_i$  generates  $\bigcap_{i \in I} G_i$ .*

*Then,  $G$  admits the presentation  $G = \langle \bigcup_{i=1}^n K_i \mid \bigcup_{i=1}^n R_i \rangle$ .*

*Proof.* Denote by  $F_G$  the free group generated by  $\bigcup_{i=1}^n G_i$  and by  $F_K$  the free group generated by  $K = \bigcup_{i=1}^n K_i$ ; we view  $F_K$  as a subgroup of  $F_G$ .

The natural group homomorphism  $\pi: F_K \rightarrow G$  is surjective, because  $G$  is generated by  $\bigcup_{i=1}^n G_i$  and each  $G_i$  is generated by  $K_i$ . Moreover, each set of relators  $R_i$  corresponds to a subset of  $\ker(\pi)$ . It remains to see that  $\ker \pi$  is contained in the normal subgroup generated by  $\bigcup_{i=1}^n R_i$ .

We take an element in  $\ker(\pi)$ , which in  $F_K$  is a word

$$w = s_1 s_2 \cdots s_m$$

such that each  $s_i$  belongs to  $K$  and  $s_1 \cdots s_m = 1$  in  $G$ . Because  $G$  admits the presentation  $G = \langle \bigcup_{i=1}^n G_i \mid R_G \rangle$ , we can write  $w$  in  $F_G$  as a product

$$w = a_1 r_1 a_1^{-1} a_2 r_2 a_2^{-1} \cdots a_l r_l a_l^{-1},$$

where all the  $a_i, r_i$  are elements of  $F_G$  and  $r_i \in R_G$ , which means by definition of  $R_G$  that  $r_i = a_i b_i c_i$  for  $a_i, b_i, c_i \in G_{j(i)}$ , that is, each  $r_i$  is a word in elements of  $G_{j(i)}$ .

The word  $w$  is equal in  $F_G$  to a reduced word, whose letters are elements of  $K$  because  $w \in F_K$ . Hence, each  $g = \bigcup_{i=1}^n G_i \setminus K$  which appears in the word  $a_1 r_1 a_1^{-1} \cdots a_l r_l a_l^{-1}$  disappears after the reduction. We can thus replace each occurrence of  $g$  with any chosen element of  $F_G$  and do not change the value of the word. We do this in the following way: if  $g \in G_i \setminus K_i$ , then we replace then  $g$  with a word with letters in  $K_i$ , which belongs to  $\pi^{-1}(\pi(g))$  (this is possible since  $K_i$  generates  $G_i$ ). If  $g$  belongs to more than one of the  $G_i$ , then we can moreover assume that the letters of the word also belong to these  $K_i$ , because of the second hypothesis.

After this replacement, we obtain an equality in  $F_K$

$$s_1 \cdots s_m = b_1 t_1 b_1^{-1} b_2 t_2 b_2^{-1} \cdots b_l t_l b_l^{-1},$$

where each  $t_i$  is a word with letters in  $K_{j(i)}$ , such that  $\pi(t_i) = 1$ . For each  $i = 1, \dots, n$  denote by  $F_{K_i}$  the free group generated by  $K_i$  and by  $\pi_i : F_{K_i} \rightarrow G_i$  the natural group homomorphism onto  $G_i$  whose kernel is generated by  $R_i$ . We consider  $F_{K_i}$  as a subgroup of  $F_K$ , which means that  $\pi_i = \pi|_{F_{K_i}}$ , and hence  $\ker(\pi_i) = \ker(\pi) \cap F_{K_i}$ . Therefore,  $\pi_i(t_i) = 1$ , and thus  $t_i$  is a product of conjugates of  $R_{j(i)}$ . This yields the result.  $\square$

**COROLLARY 6.7.** *Let  $K \subset \text{Aut}(\mathbb{P}^2)$ ,  $K_0 \subset \text{Aut}(\mathbb{F}_0)$ ,  $K_2 \subset \text{Aut}(\mathbb{F}_2)$  be compact neighbourhoods of 1 in the respective groups. Then  $\text{Bir}(\mathbb{P}^2)$  is compactly presented by  $K \cup K_0 \cup K_2 \cup \{\tau_{12}\}$ .*

*Proof.* Lemma 6.6 yields that the union of any compact generating sets of  $\text{Aut}(\mathbb{P}^2)$ ,  $\text{Aut}(\mathbb{F}_0)$ ,  $\text{Aut}(\mathbb{F}_2)$  giving a compact presentation of the respective groups yields a compact presentation of  $\mathfrak{G}$ , the generalized amalgamated product of  $\text{Aut}(\mathbb{P}^2)$ ,  $\text{Aut}(\mathbb{F}_0)$ ,  $\text{Aut}(\mathbb{F}_2)$  along their pairwise intersection divided by one relation. Such compact generating sets are given by Corollary 6.5: Any compact neighbourhood of 1 of the groups  $\text{Aut}(\mathbb{P}^2)$  and  $\text{Aut}(\mathbb{F}_2)$ , respectively, and the union of  $\tau_{12}$  and any compact neighbourhood of 1 in  $\text{Aut}(\mathbb{F}_0)$ . Since  $\text{Bir}(\mathbb{P}^2)$  and  $\mathfrak{G}$  are isomorphic (Theorem 5.5), the claim follows.  $\square$

**COROLLARY 6.8.** *Let  $K \subset \text{Aut}(\mathbb{P}^2)$ ,  $K_0 \subset \text{Aut}(\mathbb{F}_0)$ ,  $K_2 \subset \text{Aut}(\mathbb{F}_2)$  be compact neighbourhoods of 1 in the respective groups. Then  $\text{Bir}(\mathbb{P}^2)$  is compactly presented by  $K \cup K_0 \cup K_2$ .*

*Proof.* We define  $S_1 := K \cup K_0 \cup K_2$  and  $S_2 := K \cup K_0 \cup K_2 \cup \{\tau_{12}\}$ . The set  $S_2$  generates  $\text{Bir}(\mathbb{P}^2)$  by Corollary 6.7. The set  $K$  generates  $\text{Aut}(\mathbb{P}^2)$  (Corollary 6.5), hence there exists  $n \in \mathbb{N}$  such that  $\tau_{12} \in K^n$ . It follows that also  $S_2$  generates  $\text{Bir}(\mathbb{P}^2)$  and moreover that  $S_1 \subset S_2 \subset (S_1)^n$ . The claim now follows from Lemma 6.2(i) and Corollary 6.7.  $\square$

Lemma 6.2(i) and Corollary 6.8 imply that to prove Theorem A (Corollary 6.10), we only need to check that, for any compact neighbourhood  $K \subset \text{Aut}(\mathbb{P}^2)$  of 1, there exist  $K_i \subset \text{Aut}(\mathbb{F}_i)$ ,  $i = 0, 2$ , compact neighbourhoods of 1 and integers  $m, m', n \in \mathbb{N}$  such that  $(K \cup \{\sigma_3\})^m \subset (K \cup K_0 \cup K_2)^n \subset (K \cup \{\sigma_3\})^{m'}$ .

**LEMMA 6.9.** *Let  $K \subset \text{Aut}(\mathbb{P}^2)$  be a compact neighbourhood of 1. Then there exists  $N \in \mathbb{N}$  such that  $(K \cup \{\sigma_3\})^N$  contains compact neighbourhoods of 1 in  $\text{Aut}(\mathbb{F}_i)$  for  $i = 0, 2$ .*

*Proof.* Let  $\mathcal{A}_0 = \text{Aut}(\mathbb{F}_0)^0 \cap \text{Aut}(\mathbb{P}^2)$  and  $\mathcal{A}_2 = \text{Aut}(\mathbb{F}_2) \cap \text{Aut}(\mathbb{P}^2)$ , which are connected algebraic subgroups of  $\text{Aut}(\mathbb{F}_0)$  and  $\text{Aut}(\mathbb{F}_2)$ , respectively (Lemma 2.4). For  $i = 0, 2$ , the set  $K_i = K \cap \mathcal{A}_i$  is a compact neighbourhood of 1 in  $\mathcal{A}_i$ . Corollary 6.5 implies that  $\mathcal{A}_i =$

$\bigcup_{n \in \mathbb{N}} (K_i)^n$  for  $i = 0, 2$ . It follows that

$$\mathcal{A}_0 \sigma_3 \mathcal{A}_0 = \bigcup_{n \in \mathbb{N}} (K_0)^n \sigma_3 (K_0)^n, \quad \mathcal{A}_2 \sigma_2 \mathcal{A}_2 = \bigcup_{n \in \mathbb{N}} (K_2)^n \sigma_2 (K_2)^n.$$

The sets  $\mathcal{A}_0 \sigma_3 \mathcal{A}_0$  and  $\mathcal{A}_2 \sigma_2 \mathcal{A}_2$  are Zariski-open subsets of  $\text{Aut}(\mathbb{F}_0)$  and  $\text{Aut}(\mathbb{F}_2)$ , respectively (Lemma 2.4), and are thus locally compact, and hence Baire spaces. There exists then some  $m \in \mathbb{N}$  such that  $(K_0)^m \sigma_3 (K_0)^m$  and  $(K_2)^m \sigma_2 (K_2)^m$  have non-empty interior in  $\mathcal{A}_0 \sigma_3 \mathcal{A}_0$  and  $\mathcal{A}_2 \sigma_2 \mathcal{A}_2$ , and thus in  $\text{Aut}(\mathbb{F}_0)$  and  $\text{Aut}(\mathbb{F}_2)$ , respectively.

Since  $(K_i)^m \sigma_j (K_i)^m \subset (K_i \cup \{\sigma_j\})^{2m+1}$ , the sets  $(K_0 \cup \{\sigma_3\})^{2m+1}$  and  $(K_2 \cup \{\sigma_2\})^{2m+1}$  also have non-empty interior in  $\text{Aut}(\mathbb{F}_0)$  and  $\text{Aut}(\mathbb{F}_2)$ , respectively. Since  $(K_i)^{-1} \subset K_i^{m_i}$  for some big  $m_i$  and  $(\sigma_j)^{-1} = \sigma_j$ , the sets  $(K_0 \cup \{\sigma_3\})^{m'}$  and  $(K_2 \cup \{\sigma_2\})^{m'}$  are neighbourhoods of 1 in the corresponding groups for some  $m'$  big enough. Since  $\text{Bir}(\mathbb{P}^2)$  is generated by  $K \cup \{\sigma_3\}$  (by the Noether–Castelnuovo theorem), we find  $m''$  such that  $\sigma_2 \in (K \cup \{\sigma_3\})^{m''}$ . A suitable power of  $K \cup \{\sigma_3\}$  contains thus  $(K_0 \cup \{\sigma_3\})^{m'}$  and  $(K_2 \cup \{\sigma_2\})^{m'}$ .  $\square$

**COROLLARY 6.10** (Theorem A). *Let  $K \subset \text{Aut}(\mathbb{P}^2)$  be a compact neighbourhood of 1. Then  $\text{Bir}(\mathbb{P}^2)$  is compactly presented by  $K \cup \{\sigma_3\}$ .*

*Proof.* According to Lemma 6.9, there exist  $N \in \mathbb{N}$  and compact neighbourhoods  $K_0, K_2$  of 1 in  $\text{Aut}(\mathbb{F}_0)$  and  $\text{Aut}(\mathbb{F}_2)$ , respectively, such that  $(K \cup \{\sigma_3\})^N$  contains  $K_0 \cup K_2$ .

We define  $S_1 := K \cup \{\sigma_3\}$  and  $S_2 := K \cup K_0 \cup K_2$ . Because  $\sigma_3 \in \text{Aut}(\mathbb{F}_0)^0$  and  $\text{Aut}(\mathbb{F}_0)^0$  is compactly generated by  $K_0$  (Lemma 6.2(ii)), there exists  $M \in \mathbb{N}$  such that  $\sigma_3 \in (K_0)^M$ . It follows that  $S_1 \subset (S_2)^M$ . Since  $S_2 \subset (S_1)^N$ , we have  $S_1 \subset (S_2)^M \subset (S_1)^{MN}$ . The group  $\text{Bir}(\mathbb{P}^2)$  being compactly presented by  $S_2$  (Corollary 6.8), it is also compactly presented by  $S_1$  (Lemma 6.2(i)).  $\square$

*Acknowledgements.* The author would like to thank Jérémy Blanc, Yves de Cornulier and Pierre de la Harpe for interesting and helpful discussions.

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